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Bidemocratic Bases and Their Connections with Other Greedy-Type Bases

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Abstract

In nonlinear greedy approximation theory, bidemocratic bases have traditionally played the role of dualizing democratic, greedy, quasi-greedy, or almost greedy bases. In this article we shift the viewpoint and study them for their own sake, just as we would with any other kind of greedy-type bases. In particular we show that bidemocratic bases need not be quasi-greedy, despite the fact that they retain a strong unconditionality flavor which brings them very close to being quasi-greedy. Our constructive approach gives that for each $1 < p < \infty$ the space ℓ_p has a bidemocratic basis which is not quasi-greedy. We also present a novel method for constructing conditional quasi-greedy bases which are bidemocratic, and provide a characterization of bidemocratic bases in terms of the new concepts of truncation quasi-greediness and partially democratic bases.

Keywords Nonlinear approximation · Thresholding greedy algorithm · Quasi-greedy basis · Democracy

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1 Introduction and Background

Let $\mathbb{X}$ be an infinite-dimensional separable Banach space (or, more generally, a quasi-Banach space) over the real or complex field $\mathbb{F}$. Throughout this paper, unless otherwise stated, by a *basis* of $\mathbb{X}$ we mean a norm-bounded sequence $\mathcal{X} = (\boldsymbol{x}_n)_{n=1}^{\infty}$ that generates the entire space, in the sense that

$$\overline{\operatorname{span}}(\boldsymbol{x}_n : n \in \mathbb{N}) = \mathbb{X},$$

and for which there is a (unique) norm-bounded sequence $\mathcal{X}^* = (\boldsymbol{x}_n^*)_{n=1}^{\infty}$ in the dual space $\mathbb{X}^*$ such that $(\boldsymbol{x}_n, \boldsymbol{x}_n^*)_{n=1}^{\infty}$ is a biorthogonal system. We will refer to the basic sequence $\mathcal{X}^*$ as to the *dual basis* of $\mathcal{X}$.

We recall that the basis $\mathcal{X} = (\boldsymbol{x}_n)_{n=1}^{\infty}$ is called *democratic* if there is a constant Δ such that

$$\left\| \sum_{k\in A} \boldsymbol{x}_k \right\| \le \Delta \left\| \sum_{k\in B} \boldsymbol{x}_k \right\|$$

whenever A and B are finite subsets of $\mathbb{N}$ with $|A| \le |B|$. The *fundamental function* $\varphi \colon \mathbb{N} \to [0, \infty)$ of $\mathcal{X}$ is then defined by

$$\varphi(m) = \sup_{|A|\le m} \left\| \sum_{k\in A} \boldsymbol{x}_k \right\|, \quad m \in \mathbb{N},$$

while the *dual fundamental function* of $\mathcal{X}$ is just the fundamental function of its dual basis, i.e.,

$$\varphi^*(m) = \sup_{|A|\le m} \left\| \sum_{k\in A} \boldsymbol{x}_k^* \right\|, \quad m \in \mathbb{N}.$$

In general it is not true that if a basis $\mathcal{X} = (\boldsymbol{x}_n)_{n=1}^{\infty}$ is democratic, then its dual basis $\mathcal{X}^*$ is democratic as well. For instance, the L_1-normalized Haar system is an unconditional democratic basis of the dyadic Hardy space H_1 [27, 28], but the L_∞-normalized Haar system is not democratic in the dyadic BMO-space [24]. In order to better understand how certain greedy-like properties dualize, Dilworth et al. introduced in [16] a strengthened form of democracy. Notice that the elementary computation

$$m = \left(\sum_{k\in A} \boldsymbol{x}_k^*\right)\left(\sum_{k\in A} \boldsymbol{x}_k\right) \le \left\| \sum_{k\in A} \boldsymbol{x}_k^* \right\| \left\| \sum_{k\in A} \boldsymbol{x}_k \right\| \quad \text{if } |A| = m,$$

yields the estimate

$$m \le \varphi(m)\, \varphi^*(m), \quad m \in \mathbb{N}.$$

A basis $\mathcal{X} = (\boldsymbol{x}_n)_{n=1}^{\infty}$ is then said to be *bidemocratic* if the reverse inequality is fulfilled up to a constant, i.e., $\mathcal{X}$ is bidemocratic if there is a constant C such that

$$\varphi(m)\, \varphi^*(m) \le C\, m, \quad m \in \mathbb{N}.$$

Amongst other relevant results relative to this kind of bases in Banach spaces, Dilworth et al. showed in [16] that being quasi-greedy passes to dual bases under the assumption of bidemocracy (see [16, Theorem 5.4]). Since the dual basis of a bidemocratic basis is democratic, it follows that the corresponding result also holds for almost greedy and greedy bases. That is, if a bidemocratic basis is almost greedy (respectively, greedy), then so is its dual basis.

Despite the instrumental role played by bidemocratic bases as a key that permits dualizing some greedy-type properties, it is our contention in this paper that these bases are of interest by themselves and that they deserve to be studied as any other kind of greedy-like basis. For instance, the unconditionality constants of bidemocratic bases have been estimated (see Theorem 3.5 below), which sheds some information on the performance of the greedy algorithm when it is implemented specifically for these bases.

To undertake our task we must first place bidemocratic bases in the map by relating them with other types of bases that are relevant in the theory. In this respect the most important open question is whether bidemocratic bases are quasi-greedy. This problem is motivated by recent results that show that bidemocratic bases have uniform boundedness properties of certain (nonlinear) truncation operators that make them very close to quasi-greedy bases (see [1, Proposition 5.7]). In our language, bidemocratic bases are truncation quasi-greedy. In Sect. 3 we will solve this question in the negative by proving that bidemocracy is not in general strong enough to ensure quasi-greediness and show that for $1 < p < \infty$ the space ℓ_p has a bidemocratic basis which is not quasi-greedy.

Before that, we will look for sufficient conditions for a basis to be bidemocratic. Here one must take into account that if $\mathcal{X}$ is bidemocratic then both $\mathcal{X}$ and $\mathcal{X}^*$ are democratic but the converse fails. The first positive result we find in the literature in the reverse direction goes back to the classical monograph [22] from 1977 (way before the term democratic basis was even coined!), where Lindenstrauss and Tzafriri proved that subsymmetric bases, i.e., bases that are unconditional and equivalent to all of their subbases, are bidemocratic (see [22, Proposition 3.a.6]). Many years later, Dilworth et al. [16] proved in 2003 the aforementioned Theorem 5.4 from [16], which tells us that if $\mathcal{X}$ and $\mathcal{X}^*$ are quasi-greedy and democratic then $\mathcal{X}$ is bidemocratic. In Sect. 2 we extend this result by relaxing the conditions on the bases $\mathcal{X}$ and $\mathcal{X}^*$ while still attaining the bidemocracy of $\mathcal{X}$.

Turning to quasi-greedy bases, it is natural and consistent with our discussion in this paper, to further the study of conditional quasi-greedy bases by looking for conditional bidemocratic quasi-greedy bases, i.e., conditional almost greedy bases whose dual bases are also almost greedy. The previous methods for building conditional almost greedy bases in Banach spaces yield either bases whose fundamental function coincides with the fundamental function of the canonical basis of ℓ_1, or bases whose fundamental function increases steadily enough (formally, bases that have the upper regularity property and the lower regularity property). In the former case, the bases are not bidemocratic unless they are equivalent to the canonical ℓ_1-basis; in the latter, the bases are always bidemocratic by [16, Proposition 4.4]. The existence of conditional bidemocratic quasi-greedy bases which do not have the upper regularity property seems to be an unexplored area. In Sect. 4 we contribute to this topic by developing a new method for building bidemocratic, conditional, quasi-greedy bases with arbitrary fundamental functions.

Throughout this paper we will use standard notation and terminology from Banach spaces and greedy approximation theory, as can be found, e.g., in [6]. We also refer the reader to the recent article [1] for other more specialized notation. We next single out however the most heavily used terminology.

For broader applicability, whenever it is possible we will establish our results in the setting of quasi-Banach spaces. Let us recall that a *quasi-Banach space* is a vector space $\mathbb{X}$ over the real or complex field $\mathbb{F}$ equipped with a *quasi-norm*, i.e., a map $\|\cdot\|\colon \mathbb{X} \to [0,\infty)$ that satisfies all the usual properties of a norm with the exception of the triangle law, which is replaced with the condition

$$\|f+g\| \le \kappa(\|f\|+\|g\|), \quad f, g \in \mathbb{X}, \tag{1.1}$$

for some $\kappa \ge 1$ independent of f and g, and moreover $(\mathbb{X}, \|\cdot\|)$ is complete. The *modulus of concavity* of the quasi-norm is the smallest constant $\kappa \ge 1$ in (1.1). Given $0 < p \le 1$, a *p-Banach space* will be a quasi-Banach space whose quasi-norm is p-subadditive, i.e.,

$$\|f+g\|^p \le \|f\|^p + \|g\|^p, \quad f,\ g \in \mathbb{X}.$$

By the Aoki-Rolewicz theorem [9], every quasi-Banach space $\mathbb{X}$ is locally p-convex for some $0 < p \le 1$, i.e., $\mathbb{X}$ can be endowed with an equivalent p-subadditive quasi-norm. While the quasi-norm on a quasi-Banach space needs not be a continuous map (see [19, p. 566]), p-subadditive quasi-norms always are. For this reason there will be no loss of generality in assuming that all the quasi-Banach spaces that we will use are equipped with a continuous quasi-norm.

Some authors have studied the Thresholding Greedy Algorithm, or TGA for short, for more demanding types of bases that we will bring into play on occasion. A sequence $\mathcal{X} = (\boldsymbol{x}_n)_{n=1}^\infty$ of $\mathbb{X}$ is said to be a *Schauder basis* if for every $f \in \mathbb{X}$ there is a unique sequence $(a_n)_{n=1}^\infty$ in $\mathbb{F}$ such that $f = \sum_{n=1}^\infty a_n\, \boldsymbol{x}_n$, where the convergence of the series is understood in the topology induced by the quasi-norm. If $\mathcal{X}$ is a Schauder basis we define the biorthogonal functionals associated to $\mathcal{X}$ by $\boldsymbol{x}_k^*(f) = a_k$ for all $f = \sum_{n=1}^\infty a_n\, \boldsymbol{x}_n \in \mathbb{X}$ and $k \in \mathbb{N}$. The *partial-sum projections* $S_m\colon \mathbb{X} \to \mathbb{X}$ with respect to the Schauder basis $\mathcal{X}$, given by

$$f \mapsto S_m(f) = \sum_{n=1}^{m} \boldsymbol{x}_n^*(f)\, \boldsymbol{x}_n, \quad f \in \mathbb{X},\ m \in \mathbb{N},$$

are uniformly bounded, whence we infer that $\sup_n \|\boldsymbol{x}_n\| \, \|\boldsymbol{x}_n^*\| < \infty$. Hence, if a Schauder basis $\mathcal{X}$ is semi-normalized, i.e.,

$$0 < \inf_n \|\boldsymbol{x}_n\| \le \sup_n \|\boldsymbol{x}_n\| < \infty,$$

then $(\boldsymbol{x}_n^*)_{n=1}^\infty$ is norm-bounded and so $\mathcal{X}$ is a basis in the sense of this paper. If $\mathcal{X} = (\boldsymbol{x}_n)_{n=1}^\infty$ is a Schauder basis, then the *coefficient transform*

$$f \mapsto (\boldsymbol{x}_n^*(f))_{n=1}^\infty, \quad f \in \mathbb{X},$$

is one-to-one, that is, the basis $\mathcal{X}$ is *total*. In the case when $\|S_m\| \le 1$ for all $m \in \mathbb{N}$ the Schauder basis $\mathcal{X}$ is said to be *monotone*.

Given $A \subseteq \mathbb{N}$, we will use $\mathcal{E}_A$ to denote the set consisting of all families $(\varepsilon_n)_{n\in A}$ in $\mathbb{F}$ with $|\varepsilon_n| = 1$ for all $n \in A$. Given a basis $\mathcal{X} = (\boldsymbol{x}_n)_{n=1}^\infty$ of $\mathbb{X}$, a finite set $A \subseteq \mathbb{N}$ and $\varepsilon = (\varepsilon_n)_{n\in A} \in \mathcal{E}_A$, it is by now customary to use

$$\mathbb{1}_{\varepsilon,A}[\mathcal{X}, \mathbb{X}] = \textstyle\sum_{n\in A} \varepsilon_n\, \boldsymbol{x}_n \ \text{(resp., } \mathbb{1}^*_{\varepsilon,A}[\mathcal{X}, \mathbb{X}] = \sum_{n\in A} \varepsilon_n\, \boldsymbol{x}_n^*\text{)}.$$

If the basis and the space are clear from context we simply put $\mathbb{1}_{\varepsilon,A}$ (resp., $\mathbb{1}^*_{\varepsilon,A}$), and if $\varepsilon_n = 1$ for all $n \in A$ we put $\mathbb{1}_A$ (resp., $\mathbb{1}^*_A$). Associated with the fundamental function φ of the basis are the *upper super-democracy function* of $\mathcal{X}$,

$$\boldsymbol{\varphi_u}(m) = \boldsymbol{\varphi_u}[\mathcal{X}, \mathbb{X}](m) = \sup\left\{\left\|\mathbb{1}_{\varepsilon,A}\right\| : |A| \le m,\ \varepsilon \in \mathcal{E}_A\right\}, \quad m \in \mathbb{N},$$

and the *lower super-democracy function* of $\mathcal{X}$,

$$\boldsymbol{\varphi_l}(m) = \boldsymbol{\varphi_l}[\mathcal{X}, \mathbb{X}](m) = \inf\left\{\left\|\mathbb{1}_{\varepsilon,A}\right\| : |A| \ge m,\ \varepsilon \in \mathcal{E}_A\right\}, \quad m \in \mathbb{N}.$$

The growth of $\boldsymbol{\varphi_u}$ is of the same order as φ (see [1, inequality (8.3)]), and so the basis $\mathcal{X}$ is bidemocratic if and only if

$$\Delta_b := \sup_{m\in\mathbb{N}} \frac{1}{m} \boldsymbol{\varphi_u}[\mathcal{X}, \mathbb{X}](m) \boldsymbol{\varphi_u}[\mathcal{X}^*, \mathbb{X}^*](m) < \infty$$

(see [1, Lemma 5.5]), in which case Δ_b is called the *bidemocracy constant* of $\mathcal{X}$.

The symbol $\alpha_j \lesssim \beta_j$ for $j \in J$ means that there is a positive constant C such that the families of nonnegative real numbers $(\alpha_j)_{j\in J}$ and $(\beta_j)_{j\in J}$ are related by the inequality $\alpha_j \le C\beta_j$ for all $j \in J$. If $\alpha_j \lesssim \beta_j$ and $\beta_j \lesssim \alpha_j$ for $j \in J$ we say that $(\alpha_j)_{j\in J}$ and $(\beta_j)_{j\in J}$ are equivalent, and write $\alpha_j \approx \beta_j$ for $j \in J$.

We finally recall that a basis $\mathcal{X} = (\boldsymbol{x}_n)_{n=1}^\infty$ of a quasi-Banach space $\mathbb{X}$ is said to *dominate* another basis $\mathcal{Y} = (\boldsymbol{y}_n)_{n=1}^\infty$ of a (possibly different) quasi-Banach space $\mathbb{Y}$, if there is a bounded linear map T from $\mathbb{X}$ onto $\mathbb{Y}$ with $T(\boldsymbol{x}_n) = \boldsymbol{y}_n$ for all $n \in \mathbb{N}$. If the map T is an isomorphism, i.e., $\mathcal{X}$ dominates $\mathcal{Y}$ and vice-versa, we say that $\mathcal{X}$ and $\mathcal{Y}$ are *equivalent*.

2 From Truncation Quasi-greedy to Bidemocratic Bases

Let $\mathcal{X} = (\boldsymbol{x}_n)_{n=1}^\infty$ be a semi-normalized basis for a quasi-Banach space $\mathbb{X}$ with dual basis $(\boldsymbol{x}_n^*)_{n=1}^\infty$. For each $f \in \mathbb{X}$ and each $B \subseteq \mathbb{N}$ finite, put

$$\mathcal{U}(f, B) = \min_{n\in B} |\boldsymbol{x}_n^*(f)| \sum_{n\in B} \operatorname{sign}(\boldsymbol{x}_n^*(f))\, \boldsymbol{x}_n.$$

Given $m \in \mathbb{N} \cup \{0\}$, the *m-th-restricted truncation operator* $\mathcal{U}_m : \mathbb{X} \to \mathbb{X}$ is defined as

$$\mathcal{U}_m(f) = \mathcal{U}(f, A_m(f)), \quad f \in \mathbb{X},$$

where $A = A_m(f) \subseteq \mathbb{N}$ is a *greedy set* of f of cardinality m, i.e., $|\boldsymbol{x}_n^*(f)| \geq |\boldsymbol{x}_k^*(f)|$ whenever $n \in A$ and $k \notin A$. The set A depends on f and m, and may not be unique; if this happens we take any such set. We put

$$\Lambda_u = \Lambda_u[\mathcal{X}, \mathbb{X}] = \sup\{\|\mathcal{U}(f, B)\| : B \text{ greedy set of } f,\ \|f\| \leq 1\}.$$

If the quasi-norm is continuous, applying a perturbation technique yields

$$\Lambda_u = \sup_m \|\mathcal{U}_m\|.$$

Thus, the basis $\mathcal{X}$ is said to be *truncation quasi-greedy* if $(\mathcal{U}_m)_{m=1}^\infty$ is a uniformly bounded family of (nonlinear) operators, or equivalently, if and only if $\Lambda_u < \infty$. In this case we will refer to Λ_u as the *truncation quasi-greedy constant* of the basis.

Quasi-greedy bases are truncation quasi-greedy (see [16, Lemma 2.2] and [1, Theorem 4.13]), but the converse does not hold in general. The first case in point appeared in [15, Example 4.8], where the authors constructed a basis that dominates the unit vector system of $\ell_{1,\infty}$ (hence it is truncation quasi-greedy by [1, Proposition 9.4]) but it is not quasi-greedy. In spite of that, truncation quasi-greedy bases still enjoy most of the nice unconditionality-like properties of quasi-greedy bases. For instance, they are quasi-greedy for large coefficients (QGLC for short), suppression unconditional for constant coefficients (SUCC for short), and lattice partially unconditional (LPU for short). See [1, Sections 3 and 4] for the precise definitions and the proofs of these relations.

In turn, if $\mathcal{X}$ is bidemocratic then both $\mathcal{X}$ and its dual basis $\mathcal{X}^*$ are truncation quasi-greedy ([1, Proposition 5.7]). In this section we study the converse implication, i.e., we want to know which additional conditions make a truncation quasi-greedy basis bidemocratic. A good starting point is the following result, which uses the upper regularity property (URP for short) and which is valid only for Banach spaces. Following [16] we shall say that a basis has the URP if there is an integer $b \geq 3$ so that its fundamental function φ satisfies

$$2\varphi(bm) \leq b\varphi(m), \quad m \in \mathbb{N}. \tag{2.1}$$

Theorem 2.1 (see [1, Lemma 9.8 and Proposition 10.17(iii)]) *Let $\mathcal{X}$ be a basis of a Banach space $\mathbb{X}$. Suppose that $\mathcal{X}$ is democratic, truncation quasi-greedy, and has the URP. Then $\mathcal{X}$ is bidemocratic (and so $\mathcal{X}^*$ is truncation quasi-greedy too).*

Can we do any better? Dilworth et al. characterized quasi-greedy bidemocratic bases as those quasi-greedy bases whose dual bases are quasi-greedy and such that both the basis and its dual basis fulfil an additional condition called conservativeness ([16, Theorem 5.4]). Recall that a basis is said to be *conservative* if there is a constant C such that $\|\mathbb{1}_A\| \leq C\|\mathbb{1}_B\|$ whenever $|A| \leq |B|$ and $\max(A) \leq \min(B)$. Our objection to this concept is that it is not preserved under rearrangements of the basis. Thus, since the greedy algorithm is "reordering invariant" (i.e., if π is a permutation of $\mathbb{N}$, the greedy algorithm with respect to the bases $(\boldsymbol{x}_n)_{n=1}^\infty$ and $(\boldsymbol{x}_{\pi(n)})_{n=1}^\infty$ is the same) when working with conservative bases we are bringing an outer element into the theory. This is the reason why we establish our characterization of bidemocratic bases below in terms of a reordering invariant new class of bases which is more general than the class of conservative bases and which we next define.

Definition 2.2 We say that a basis is *partially democratic* if there is a constant C such that for each $D \subseteq \mathbb{N}$ finite there is $D \subseteq E \subseteq \mathbb{N}$ finite such that $\|\mathbb{1}_A\| \leq C\|\mathbb{1}_B\|$ whenever $A \subseteq D$ and $B \subseteq \mathbb{N}\setminus E$ satisfy $|A| \leq |B|$. In this case we will refer to the optimal constant C as the *partial democracy constant* of the basis.

Lemma 2.3 and Proposition 2.4 are well-known but we re-state them for the sake of recording the precise constants in the respective estimates. From now on we will use $\gamma = 2$ if $\mathbb{F} = \mathbb{R}$ or $\gamma = 4$ if $\mathbb{F} = \mathbb{C}$.

Lemma 2.3 (See [1, Proposition 4.16] or [4, Lemma 5.2]) *Suppose that $\mathcal{X} = (\boldsymbol{x}_n)_{n=1}^\infty$ is a truncation quasi-greedy basis of a quasi-Banach space $\mathbb{X}$. Then there is a constant C depending on the modulus of concavity of $\mathbb{X}$ and the truncation quasi-greedy constant of $\mathcal{X}$ such that*

$$\left\| \sum_{n\in A} a_n \,\boldsymbol{x}_n \right\| \leq C\|f\|$$

for all $f \in \mathbb{X}$, all $A \subseteq \mathbb{N}$ finite, and all finite families $(a_n)_{n\in A}$ such that $\max_{n\in A} |a_n| \leq \min_{n\in A} |\boldsymbol{x}_n^*(f)|$. *In the case when $\mathbb{X}$ is a p-Banach space, we can choose $C = \gamma^{1/p}(2^p - 1)^{-1/p}\Lambda_u^2$, where Λ_u is the truncation quasi-greedy constant of $\mathcal{X}$. If, in addition,* $\operatorname{sign}(a_n) = \operatorname{sign}(\boldsymbol{x}_n^*(f))$ *for all $n \in A$, we can choose $C = (2^p - 1)^{-1/p}\Lambda_u^2$.*

Proposition 2.4 (See [1, Proposition 4.19]) *Let $\mathcal{X} = (\boldsymbol{x}_n)_{n=1}^{\infty}$ be a truncation quasi-greedy basis of a Banach space $\mathbb{X}$. Then, there is a constant Λ such that for every $A \subseteq \mathbb{N}$ finite and all $\varepsilon \in \mathcal{E}_A$ there is $f^* \in \operatorname{span}(\boldsymbol{x}_n^* \colon n \in A)$ with $\|f^*\| = 1$ such that*

$$\|\mathbb{1}_{\varepsilon,A}\| \le \Lambda |f^*(\mathbb{1}_{\varepsilon,A})|.$$

In fact, the inequality holds with $\Lambda = \Lambda_u^2$.

Theorem 2.5 *Let $\mathcal{X}$ be a basis of a Banach space $\mathbb{X}$. Suppose that both $\mathcal{X}$ and $\mathcal{X}^*$ are truncation quasi-greedy and partially democratic. Then $\mathcal{X}$ is bidemocratic.*

Proof We will customize the proof of [16, Theorem 5.4] to suit our more general statement. Set $\boldsymbol{\varphi_u} = \boldsymbol{\varphi_u}[\mathcal{X}, \mathbb{X}]$ and $\boldsymbol{\varphi_u}^* = \boldsymbol{\varphi_u}[\mathcal{X}^*, \mathbb{X}^*]$. Let Δ_d and Δ_d^* be the partial democracy constants of $\mathcal{X}$ and $\mathcal{X}^*$ respectively, and let Λ_u and Λ_u^* be the truncation quasi-greedy constant of $\mathcal{X}$ and $\mathcal{X}^*$, respectively. Given $m \in \mathbb{N}$, fix $0 < \epsilon < 1$ and choose sets B_1, B_2 and signs $\varepsilon \in \mathcal{E}_{B_1}$, $\varepsilon' \in \mathcal{E}_{B_2}$ so that $|B_1| \le m$, $|B_2| \le m$, and

$$\|\mathbb{1}_{\varepsilon,B_1}\| \ge (1-\epsilon)\boldsymbol{\varphi_u}(m) \quad \text{and} \quad \|\mathbb{1}^*_{\varepsilon',B_2}\| \ge (1-\epsilon)\boldsymbol{\varphi_u}^*(m). \tag{2.2}$$

Use partial democracy to pick $D \subseteq \mathbb{N}$ disjoint with $B_1 \cup B_2$ such that $|D| = 2m$, $\|\mathbb{1}_B\| \le \Delta_d \|\mathbb{1}_A\|$, and $\|\mathbb{1}_B^*\| \le \Delta_d^* \|\mathbb{1}_A^*\|$ whenever $B \subseteq B_1 \cup B_2$ and $A \subseteq D$ satisfy $|B| \le |A|$.

Using (2.2) and partial democracy we obtain that for all $A \subseteq D$ with $|A| \ge m$,

$$(1-\epsilon)\boldsymbol{\varphi_u}(m) \le \mathbf{C}\|\mathbb{1}_A\|, \quad \text{and} \quad (1-\epsilon)\boldsymbol{\varphi_u}^*(m) \le \mathbf{C}^*\|\mathbb{1}_A^*\|, \tag{2.3}$$

where $\mathbf{C} = \gamma\Delta_d$ and $\mathbf{C}^* = \gamma\Delta_d^*$. Taking into account (2.3), Proposition 2.4 gives that for such subsets A of $\mathbb{N}$ the set

$$\mathcal{K}_A = \left\{ f^* \in \operatorname{span}(\boldsymbol{x}_n^* \colon n \in A) \colon \|f^*\| \le 1,\ f^*(\mathbb{1}_A) \ge \frac{(1-\epsilon)\boldsymbol{\varphi_u}(m)}{\mathbf{C}\Lambda_u^2} \right\}$$

is convex, closed, and nonempty. Note that $\mathcal{K}_A$ increases with A, and that for $f^* \in \mathcal{K}_A$,

$$\sum_{n\in A} |f^*(\boldsymbol{x}_n)| = f^*\left(\mathbb{1}_{\overline{\varepsilon(f^*)},A}\right) \le \|f^*\| \left\|\mathbb{1}_{\overline{\varepsilon(f^*)},A}\right\| \le \boldsymbol{\varphi_u}(|A|). \tag{2.4}$$

Pick $f^* \in \mathcal{K}_D$ that minimizes $\sum_{n\in D} |f^*(\boldsymbol{x}_n)|^2$. The geometric properties of minimizing vectors on convex subsets of Hilbert spaces yield

$$\sum_{n\in D} |f^*(\boldsymbol{x}_n)|^2 \le \Re\left(\sum_{n\in D} f^*(\boldsymbol{x}_n) g^*(\boldsymbol{x}_n)\right), \quad g^* \in \mathcal{K}_D. \tag{2.5}$$

Let E be a greedy set of f^* with $|E| = m$, and put $A = D \setminus E$. Since $\mathcal{X}^*$ is truncation quasi-greedy, by Lemma 2.3 we have

$$\min_{n\in E} |f^*(\boldsymbol{x}_n)| \, \|\mathbb{1}_E^*\| \le \gamma(\Lambda_u^*)^2 \|f^*\| \le \gamma(\Lambda_u^*)^2. \tag{2.6}$$

Pick $g^* \in \mathcal{K}_A$. Combining (2.5), (2.4), (2.6) and (2.3) gives

$$\begin{aligned}
\sum_{n\in D} |f^*(\boldsymbol{x}_n)|^2 &\le \sum_{n\in A} |f^*(\boldsymbol{x}_n)||g^*(\boldsymbol{x}_n)| \\
&\le \min_{n\in E} |f^*(\boldsymbol{x}_n)| \sum_{n\in A} |g^*(\boldsymbol{x}_n)| \\
&\le \frac{\gamma(\Lambda_u^*)^2}{\|\mathbb{1}_E^*\|} \boldsymbol{\varphi_u}(m) \\
&\le \frac{\gamma(\Lambda_u^*)^2 \mathbf{C}^*}{(1-\epsilon)\boldsymbol{\varphi_u}^*(m)} \boldsymbol{\varphi_u}(m).
\end{aligned}$$

Hence, by the Cauchy–Bunyakovsky–Schwarz inequality,

$$\begin{aligned}
(1-\epsilon)^2(\boldsymbol{\varphi_u}(m))^2 &\le \mathbf{C}^2 \Lambda_u^4 |f^*(\mathbb{1}_D)|^2 \\
&\le \mathbf{C}^2 \Lambda_u^4 \left(\sum_{n\in D} |f^*(\boldsymbol{x}_n)| \right)^2 \\
&\le 2\mathbf{C}^2 \Lambda_u^4 m \sum_{n\in D} |f^*(\boldsymbol{x}_n)|^2 \\
&\le 2m\gamma \frac{\mathbf{C}^2 \mathbf{C}^* \Lambda_u^4 (\Lambda_u^*)^2}{(1-\epsilon)} \frac{\boldsymbol{\varphi_u}(m)}{\boldsymbol{\varphi_u}^*(m)}.
\end{aligned}$$

Since ϵ is arbitrary, we obtain

$$\boldsymbol{\varphi_u}(m)\boldsymbol{\varphi_u}^*(m) \le 2\gamma^4 \Delta_d^2 \Delta_d^* \Lambda_u^4 (\Lambda_u^*)^2 m,$$

and so the basis is bidemocratic. □

We remark that Theorem 2.5 is valid only for Banach spaces, i.e., it cannot be extended when the local convexity of the space is lifted. Indeed, for $0 < p < 1$ the canonical basis $(\boldsymbol{e}_n)_{n=1}^\infty$ of ℓ_p is not bidemocratic despite the fact that $(\boldsymbol{e}_n)_{n=1}^\infty$ is democratic and unconditional and its dual basis (the standard unit vector basis of c_0) is also democratic. Theorem 2.5 provides a characterization of bidemocratic bases in Banach spaces in terms of truncation quasi-greedy and partially democratic bases. To be precise, we have the following.

Corollary 2.6 (cf. [16, Theorem 5.4]) *Let $\mathcal{X}$ be a basis of a Banach space $\mathbb{X}$. The following are equivalent:*

(i) $\mathcal{X}$ is bidemocratic.
(ii) $\mathcal{X}$ and $\mathcal{X}^*$ are bidemocratic.
(iii) $\mathcal{X}$ and $\mathcal{X}^*$ are truncation quasi-greedy and superdemocratic.
(iv) $\mathcal{X}$ and $\mathcal{X}^*$ are truncation quasi-greedy and democratic.
(v) $\mathcal{X}$ and $\mathcal{X}^*$ are truncation quasi-greedy and conservative.
(vi) $\mathcal{X}$ and $\mathcal{X}^*$ are truncation quasi-greedy and partially democratic.

Proof The implication (i) $\Longrightarrow$ (ii) follows from [1, Lemma 5.6], while the implication (ii) $\Longrightarrow$ (iii) follows from combining [1, Proposition 5.7] with [16, Proposition 4.2]. The chain of implications (iii) $\Longrightarrow$ (iv) $\Longrightarrow$ (v) $\Longrightarrow$ (vi) is immediate. Our contribution here is the implication (vi) $\Longrightarrow$ (i), which is precisely Theorem 2.5. $\square$

3 Existence of Bidemocratic Non-quasi-greedy Bases

This section is geared towards proving the existence of bidemocratic bases which are not quasi-greedy. To that end, let us first set the minimum requirements on terminology we need for this section.

Suppose $\mathcal{X} = (\boldsymbol{x}_n)_{n=1}^{\infty}$ is a democratic basis of a quasi-Banach space $\mathbb{X}$. We shall say that $\mathcal{X}$ has the *lower regularity property* (LRP for short) if there is an integer $b \geq 2$ such

$$2\varphi(m) \leq \varphi(bm), \quad m \in \mathbb{N}. \tag{3.1}$$

In a sense, the LRP is the dual property of the URP. Abusing the language we will say that a sequence has the URP (respectively, LRP), if its terms verify the condition (2.1) (respectively, (3.1)). Note that $(\varphi(m))_{m=1}^{\infty}$ has the LRP if and only if $(m/\varphi(m))_{m=1}^{\infty}$ has the URP. If $(\varphi(m))_{m=1}^{\infty}$ has the LRP then there is $a > 0$ and $C \geq 1$ such that

$$\frac{m^a}{n^a} \leq C \frac{\varphi(m)}{\varphi(n)}, \quad n \leq m. \tag{3.2}$$

In the case when φ is non-decreasing and the sequence $(\varphi(m)/m)_{m=1}^{\infty}$ is non-increasing, φ has the LRP if and only if the weight $\boldsymbol{w} = (w_n)_{n=1}^{\infty}$ defined by $w_n = \varphi(n)/n$ is a *regular* weight, i.e., it satisfies the Dini condition

$$\sup_n \frac{1}{n w_n} \sum_{k=1}^{n} w_k < \infty$$

(see [1, Lemma 9.8]), in which case

$$\sum_{n=1}^{m} \frac{\varphi(n)}{n} \approx \varphi(m), \quad m \in \mathbb{N}. \tag{3.3}$$

For instance, the power sequence $(m^{1/p})_{m=1}^{\infty}$ has the URP if and only if $1 < p \leq \infty$, and has the LRP for all $0 < p < \infty$. Consequently, the weight $\boldsymbol{w} = (n^{-a})_{n=1}^{\infty}$ is regular for all $0 < a < 1$.

We will need the following elementary lemma involving the *harmonic numbers*

$$H_m = \sum_{n=1}^{m} \frac{1}{n}, \quad m \in \mathbb{N} \cup \{0\}.$$

Lemma 3.1 *For each* $0 < a < 1$ *there exists a constant* $C(a)$ *such that*

$$S(a, r, t) := \sum_{k=r+1}^{t} k^{-a}(k-r)^{a-1} \le C(a)(H_t - H_r), \quad t \ge 2r.$$

Proof The inequality is trivial for $r = 0$, so we assume that $r \ge 1$. If we define $f : [1, \infty) \to [0, \infty)$ by $f(u) = u^{-a}(u-1)^{a-1}$, we have

$$k^{-a}(k-r)^{a-1} \le x^{-a}(x-r)^{a-1} = \frac{1}{r} f\left(\frac{x}{r}\right), \quad k \in \mathbb{N}, \; x \in [k-1, k].$$

Hence,

$$S(a, r, t) \le \int_r^t f\left(\frac{x}{r}\right) \frac{dx}{r} = \int_1^{t/r} f(u)\, du.$$

Since f is integrable on $[1, 2]$ and $f(u) \lesssim 1/u$ for $u \in [2, \infty)$, there is a constant C_1 such that $S(a, r, t) \le C_1 \log(t/r)$. Taking into account that $H_t - H_r \ge (t-r)/t \ge 1/2$, and that there is a constant C_2 such that $\log m \le H_m \le \log m + C_2$ for all $m \in \mathbb{N}$ we are done. □

For further reference, we record an easy lemma that we will use several times. Note that it applies in particular to the harmonic series.

Lemma 3.2 *Let* $\sum_{n=1}^{\infty} c_n$ *be a divergent series of nonnegative terms. Suppose that* $\lim_n c_n = 0$. *Then, for every* $m \in \mathbb{N} \cup \{0\}$ *and* $0 \le a < b$, *there are* $m \le r < s$ *such that* $a \le \sum_{n=r+1}^{s} c_n < b$.

We will also use the following well-known lemma. Note that it could be used to prove the divergence of the harmonic series.

Lemma 3.3 (See [25, Exercise 11, p. 84]) *Let* $\sum_{n=1}^{\infty} c_n$ *be a divergent series of nonnegative terms. Then the (smaller) series*

$$\sum_{n=1}^{\infty} \frac{c_n}{\sum_{k=1}^{n} c_k}$$

also diverges.

Lorentz sequence spaces $d_{1,q}(\boldsymbol{w})$ play a relevant role in the qualitative study of greedy-like bases. Let $\boldsymbol{w} = (w_n)_{n=1}^{\infty}$ be a weight (i.e., a sequence of nonnegative numbers with $w_1 > 0$) whose primitive weight $(s_m)_{m=1}^{\infty}$, defined by $s_m = \sum_{n=1}^{m} w_n$, is unbounded and *doubling*, i.e.,

$$\sup_m \frac{s_{2m}}{s_m} < \infty.$$

Given $0 < q \le \infty$, we will denote by $d_{1,q}(\boldsymbol{w})$ the quasi-Banach space of all $f \in c_0$ whose non-increasing rearrangement $(a_n)_{n=1}^\infty$ satisfies

$$\|f\|_{d_{1,q}(\boldsymbol{w})} = \left(\sum_{n=1}^\infty a_n^q s_n^{q-1} w_n\right)^{1/q} < \infty,$$

with the usual modification if $q = \infty$. For power weights this definition yields the classical Lorentz sequence spaces $\ell_{p,q}$. To be precise, if $\boldsymbol{w} = (n^{1/p-1})_{n=1}^\infty$ for some $0 < p < \infty$, then, up to equivalence of quasi-norms, $d_{1,q}(\boldsymbol{w}) = \ell_{p,q}$, and if $(a_n)_{n=1}^\infty$ is the non-increasing rearrangement of $f \in c_0$,

$$\|f\|_{\ell_{p,q}} = \left(\sum_{n=1}^\infty a_n^q n^{q/p-1}\right)^{1/q}.$$

For a quick introduction to Lorentz sequence spaces we refer the reader to [1, Section 9.2]. Here we gather the properties of these spaces that are most pertinent for our purposes. Although it is customary to designate them after the weight $\boldsymbol{w}$, it must be conceded that as a matter of fact they depend on its primitive weight $(s_m)_{m=1}^\infty$ rather than on $\boldsymbol{w}$. That is, given weights $\boldsymbol{w} = (w_n)_{n=1}^\infty$ and $\boldsymbol{w}' = (w_n')_{n=1}^\infty$ with primitive weights $(s_m)_{m=1}^\infty$ and $(s_m')_{m=1}^\infty$, we have $d_{1,q}(\boldsymbol{w}) = d_{1,q}(\boldsymbol{w}')$ (up to equivalence of quasi-norms) if and only if $s_m \approx s_m'$ for $m \in \mathbb{N}$. The fundamental function of the unit vector system of $d_{1,q}(\boldsymbol{w})$ is equivalent to $(s_m)_{m=1}^\infty$, hence essentially it does not depend on q. We have

$$d_{1,p}(\boldsymbol{w}) \subseteq d_{1,q}(\boldsymbol{w}), \quad 0 < p < q \le \infty.$$

To show that this inclusion is strict we can for instance use the sequence

$$H_m[\boldsymbol{w}] = \sum_{n=1}^m \frac{w_n}{s_n}, \quad m \in \mathbb{N},$$

and notice that $\lim_m H_m[\boldsymbol{w}] = \infty$ by Lemma 3.3, and

$$\left\|\sum_{n=1}^m \frac{1}{s_n} \boldsymbol{e}_n\right\|_{d_{1,q}(\boldsymbol{w})} = (H_m[\boldsymbol{w}])^{1/q}, \quad m \in \mathbb{N}, \ 0 < q < \infty. \tag{3.4}$$

For $0 < q < 1$, the quasi-Banach space $d_{1,q}(\boldsymbol{w})$ is locally q-convex. In the case when $q \ge 1$, the space $d_{1,q}(\boldsymbol{w})$ is locally r-convex for all $r < 1$ but it is not locally convex in general. It is worthwhile mentioning that imposing some regularity to the primitive weight $(s_m)_{m=1}^\infty$ makes a difference. In fact, if $(s_m)_{m=1}^\infty$ has the URP then $d_{1,q}(\boldsymbol{w})$ is locally convex for all $1 \le q \le \infty$. The following lemma shows that the LRP is also of interest when dealing with Lorentz sequence spaces.

Lemma 3.4 *Let $0 < q \le \infty$, and let $(s_m)_{m=1}^\infty$ be the primitive weight of a weight $\boldsymbol{w}$. Suppose that $(s_m)_{m=1}^\infty$ has the LRP and that the weight $\boldsymbol{w}' = (w_n')_{n=1}^\infty$ given by $w_n' = s_n/n$ is non-increasing. Then:*

(i) $d_{1,q}(\boldsymbol{w}) = d_{1,q}(\boldsymbol{w}')$;
(ii) *for* $0 \le r \le t < \infty$, $H_t[\boldsymbol{w}'] - H_r[\boldsymbol{w}'] \approx H_t - H_r$ *and*
(iii) $A(r,t) := \left\| \sum_{n=r+1}^t s_n^{-1}\, \boldsymbol{e}_n \right\|_{d_{1,q}(\boldsymbol{w})} \lesssim \max\{1, (H_t - H_r)^{1/q}\}$.

Proof The first part follows from (3.3). Let $(s_m')_{m=1}^\infty$ be the primitive weight of $\boldsymbol{w}'$. The equivalence (3.3) also yields

$$\frac{w_n'}{s_n'} \approx \frac{1}{n}, \quad n \in \mathbb{N}.$$

Hence, (ii) holds. Pick $0 < a < 1/q$ such that (3.2) holds. On one hand, if $t \le 2r+1$,

$$A(r,t) \le \frac{1}{s_{r+1}} \left\| \sum_{n=r+1}^{t} \boldsymbol{e}_n \right\|_{d_{1,q}(\boldsymbol{w})} \lesssim \frac{s_{t-r}}{s_{r+1}} \le 1.$$

On the other hand, if $t \ge 2r$ using again (i) we obtain

$$A(r,t) \approx \left(\sum_{k=r+1}^{t} \frac{s_{k-r}^q}{s_k^q (k-r)} \right)^{1/q} \lesssim \left(\sum_{k=r+1}^{t} \frac{(k-r)^{aq}}{k^{aq}(k-r)} \right)^{1/q}.$$

Hence, applying Lemma 3.1 yields the desired inequality. □

To contextualize the assumptions in Theorem 3.6 below we must take into account that any basis $\mathcal{X}$ of an r-Banach space $\mathbb{X}$, $0 < r \le 1$, is dominated by the unit vector basis of the Lorentz sequence space $d_{1,r}(\boldsymbol{w})$, where the primitive weight of $\boldsymbol{w}$ is $\boldsymbol{\varphi_u}[\mathcal{X}, \mathbb{X}]$. Although it is not central in our study, in the proof of Theorem 3.6 we will keep track of the *quasi-greedy parameters* of the basis,

$$\overline{\boldsymbol{g}}_m[\mathcal{X}, \mathbb{X}] = \sup\{\|S_A[\mathcal{X}, \mathbb{X}](f)\| \colon A \text{ greedy set of } f \in B_{\mathbb{X}},\ |A| = m\},$$

where for a finite subset $A \subseteq \mathbb{N}$, we let $S_A = S_A[\mathcal{X}, \mathbb{X}] \colon \mathbb{X} \to \mathbb{X}$ denote the coordinate projection on A, i.e.,

$$S_A(f) = \sum_{n \in A} \boldsymbol{x}_n^*(f)\, \boldsymbol{x}_n, \quad f \in \mathbb{X}.$$

The quasi-greedy parameters are bounded above by the *unconditionality parameters*

$$\boldsymbol{k}_m = \boldsymbol{k}_m[\mathcal{X}, \mathbb{X}] := \sup_{|A|=m} \|S_A\|, \quad m \in \mathbb{N},$$

which are used to quantify how far the basis is from being unconditional. Thus, the following result exhibits that bidemocratic bases are close to being quasi-greedy.

Theorem 3.5 *Let $\mathbb{X}$ be a p-Banach space, $0 \le p \le 1$. If $\mathcal{X}$ is a truncation quasi-greedy basis of of $\mathbb{X}$, then*

$$\boldsymbol{k}_m[\mathcal{X}, \mathbb{X}] \lesssim (\log m)^{1/p}, \quad m \ge 2.$$

In particular, the estimate holds if $\mathcal{X}$ is bidemocratic.

Proof The first part is [4, Theorem 5.1], and we can obtain the result for bidemocratic bases applying [1, Proposition 5.7]. □

Since $(\overline{\boldsymbol{g}}_m)_{m=1}^\infty$ needs not be non-decreasing (see [23, Proposition 3.1]), we also set

$$\boldsymbol{g}_m = \boldsymbol{g}_m[\mathcal{X}, \mathbb{X}] = \sup_{k \le m} \overline{\boldsymbol{g}}_k.$$

Of course, $\mathcal{X}$ is quasi-greedy if and only if $\sup_m \boldsymbol{g}_m = \sup_m \overline{\boldsymbol{g}}_m < \infty$, and $\mathcal{X}$ is unconditional if and only if $\sup_m \boldsymbol{k}_m < \infty$.

We will use the fact that quasi-greedy bases are in particular total bases (see [1, Corollary 4.5]) to prove the advertised existence of bidemocratic non-quasi-greedy bases.

Theorem 3.6 *Let $1 < q < \infty$, and let $\boldsymbol{w} = (w_n)_{n=1}^\infty$ be a non-increasing weight whose primitive weight $(s_m)_{m=1}^\infty$ is unbounded. Let $\mathbb{X}$ be a quasi-Banach space with a basis $\mathcal{X}$. Suppose that $\mathcal{X}$ is bidemocratic with $\boldsymbol{\varphi_u}[\mathcal{X}, \mathbb{X}](m) \approx s_m$ for $m \in \mathbb{N}$, and that $\mathcal{X}$ has a subsequence dominated by the unit vector basis of $d_{1,q}(\boldsymbol{w})$. Then $\mathbb{X}$ has a non-total bidemocratic basis $\mathcal{Y}$ with*

$$\boldsymbol{\varphi_u}[\mathcal{Y}, \mathbb{X}](m) \approx s_m, \quad m \in \mathbb{N}.$$

Moreover, if $(s_m)_{m=1}^\infty$ has the LRP,

$$\overline{\boldsymbol{g}}_m[\mathcal{Y}, \mathbb{X}] \gtrsim (\log m)^{1/q'}, \quad m \ge 2,$$

where $1/q + 1/q' = 1$.

Proof Choose a subsequence $\left(\boldsymbol{x}_{\eta(k)}\right)_{k=1}^\infty$ of $\mathcal{X} = (\boldsymbol{x}_n)_{n=1}^\infty$ so that $\eta(1) \ge 2$. The linear operator $T : d_{1,q}(\boldsymbol{w}) \to \mathbb{X}$ given by

$$T(\boldsymbol{e}_k) = \boldsymbol{x}_{\eta(k)}, \quad k \in \mathbb{N},$$

is bounded. With the aid of T we proceed to perturb the basis $\mathcal{X}$ in such a way that it loses its totality while preserving bidemocracy. For each $n \in \mathbb{N}$, $n \ge 2$, define $\boldsymbol{y}_n = \boldsymbol{x}_n + \boldsymbol{z}_n$, where

$$\boldsymbol{z}_n = \begin{cases} w_k\, \boldsymbol{x}_1 & \text{if } n = \eta(k), \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that $(\boldsymbol{y}_n, \boldsymbol{x}_n^*)_{n=2}^\infty$ is a biorthogonal system. Moreover, $\mathcal{Y} := \left(\boldsymbol{y}_n\right)_{n=2}^\infty$ and $\mathcal{Y}^* := (\boldsymbol{x}_n^*)_{n=2}^\infty$ are norm-bounded because $\mathcal{X}$, $\mathcal{X}^*$, and $\mathcal{Z} = (\boldsymbol{z}_n)_{n=2}^\infty$ are. Thus, in order to prove that $\mathcal{Y}$ is a basis of $\mathbb{X}$ with dual basis $\mathcal{Y}^*$ it suffices to prove that $\boldsymbol{x}_1$ belongs to the closed linear span of $\mathcal{Y}$. To that end, we note that for each $m \in \mathbb{N}$ we have

$$\begin{aligned} f_m &:= \frac{1}{H_m[\boldsymbol{w}]} \sum_{k=1}^m \frac{1}{s_k} \boldsymbol{y}_{\eta(k)} \\ &= \boldsymbol{x}_1 + \frac{1}{H_m[\boldsymbol{w}]} \sum_{k=1}^m \frac{1}{s_k} \boldsymbol{x}_{\eta(k)} \\ &= \boldsymbol{x}_1 + \frac{1}{H_m[\boldsymbol{w}]} T(g_m), \end{aligned}$$

where $g_m = \sum_{k=1}^m s_k^{-1} \boldsymbol{e}_k$. By (3.4),

$$\| f_m - \boldsymbol{x}_1 \| \le \|T\| (H_m[\boldsymbol{w}])^{-1/q'}, \quad m \in \mathbb{N}.$$

Since by Lemma 3.3, $\lim_m H_m[\boldsymbol{w}] = \infty$ we obtain that $\lim_m f_m = \boldsymbol{x}_1$.

Since $\boldsymbol{y}_n^*(\boldsymbol{x}_1) = 0$ for all $n \ge 2$, $\mathcal{Y}$ is not a total basis. In order to prove that it is bidemocratic, we must show that $\boldsymbol{\varphi_u}[\mathcal{Y}, \mathbb{X}](m) \lesssim s_m$ and $\boldsymbol{\varphi_u}[\mathcal{Y}^*, \mathbb{X}^*](m) \lesssim m/s_m$ for $m \in \mathbb{N}$. The latter inequality is a ready consequence of the estimate $\boldsymbol{\varphi_u}[\mathcal{X}^*, \mathbb{X}^*](m) \lesssim m/s_m$ for $m \in \mathbb{N}$, which holds because $\mathcal{X}$ is bidemocratic. To prove the former, we note that, since $\boldsymbol{w}$ is non-increasing,

$$\boldsymbol{\varphi_u}[\mathcal{Z}, \mathbb{X}](m) = \|\boldsymbol{x}_1\| \sum_{k=1}^m w_k \approx s_m, \quad m \in \mathbb{N}.$$

Consequently,

$$\boldsymbol{\varphi_u}[\mathcal{Y}, \mathbb{X}](m) \lesssim \boldsymbol{\varphi_u}[\mathcal{X}, \mathbb{X}](m) + \boldsymbol{\varphi_u}[\mathcal{Z}, \mathbb{X}](m) \lesssim s_m, \quad m \in \mathbb{N}.$$

To estimate the quasi-greedy parameters in the case when $(s_m)_{m=1}^\infty$ has the LRP, we appeal to Lemma 3.2 to pick for each $m \ge 2$ natural numbers $r = r(m)$ and $s = s(m)$ with $m \le r \le s$, and

$$H_m[\boldsymbol{w}] \le H_s[\boldsymbol{w}] - H_r[\boldsymbol{w}] \le (H_m[\boldsymbol{w}])^{1/q} + H_m[\boldsymbol{w}]. \tag{3.5}$$

Moreover, since $(s_n/n)_{n=1}^\infty$ is non-increasing, thanks to parts (i) and (ii) of Lemma 3.4 we may assume without loss of generality that

$$H_t[\boldsymbol{w}] - H_r[\boldsymbol{w}] \approx H_t - H_r, \quad 0 \le r \le t. \tag{3.6}$$

Set $h_m = \sum_{k=r+1}^{s} s_k^{-1} \boldsymbol{e}_k$ and

$$\begin{aligned} u_m &= \frac{1}{H_m[\boldsymbol{w}]} \left(\sum_{k=1}^{m} \frac{1}{s_k} \boldsymbol{y}_{\eta(k)} - \sum_{k=r+1}^{s} \frac{1}{s_k} \boldsymbol{y}_{\eta(k)} \right) \\ &= \frac{1}{H_m[\boldsymbol{w}]} \left(T(g_m) - T(h_m) + (H_m[\boldsymbol{w}] - H_s[\boldsymbol{w}] + H_r[\boldsymbol{w}]) \boldsymbol{x}_1 \right). \end{aligned}$$

By Lemma 3.4 (iii), (3.4), (3.6) and (3.5),

$$\max\{\|g_m\|, \|h_m\|, |H_s[\boldsymbol{w}] - H_r[\boldsymbol{w}] - H_m[\boldsymbol{w}]|\} \lesssim H_m^{1/q}, \quad m \in \mathbb{N}.$$

Hence, $\|u_m\| \lesssim H_m^{-1/q'}$ for $m \in \mathbb{N}$. Since $A_m := \{\eta(1), \dots, \eta(m)\}$ is a greedy set of u_m with respect to $\mathcal{Y}$, and

$$\|S_{A_m}[\mathcal{Y}, \mathbb{X}](u_m)\| = \|f_m\| \approx 1, \quad m \in \mathbb{N},$$

we are done. □

Corollary 3.7 *Let $\mathbb{X}$ be a Banach space with a Schauder basis. Suppose that $\mathbb{X}$ has a complemented subspace isomorphic to $\ell_{p,q}$, where $p, q \in (1, \infty)$. Then $\mathbb{X}$ has a non-total bidemocratic basis $\mathcal{Y}$ with*

$$\boldsymbol{\varphi_u}[\mathcal{Y}, \mathbb{X}](m) \approx m^{1/p}, \quad m \in \mathbb{N},$$

and

$$\overline{\boldsymbol{g}}_m[\mathcal{Y}, \mathbb{X}] \gtrsim (\log m)^{1/q'}, \quad m \geq 2.$$

Proof An application of the Dilworth-Kalton-Kutzarova method, or DKK-method for short (see [2, 15]), yields a bidemocratic Schauder basis of $\mathbb{X}$ with fundamental function equivalent to $(m^{1/p})_{m=1}^{\infty}$ (see [2]). The direct sum of this basis with the unit vector system of $\ell_{p,q}$ is a bidemocratic Schauder basis of $\mathbb{X} \oplus \ell_{p,q} \approx \mathbb{X}$ that possesses a subsequence equivalent to the unit vector basis of $\ell_{p,q}$. Applying Theorem 3.6 we are done. □

Note that Corollary 3.7 can be applied with $1 < p = q < \infty$, so that $\ell_{p,q} = \ell_p$. Hence as a consequence we obtain the result that we announced in the Introduction.

Theorem 3.8 *Let $1 < p < \infty$. Then ℓ_p has a bidemocratic non-total (hence, non-quasi-greedy) basis.*

Theorem 3.8 leads us naturally to the question about the existence of bidemocratic non-total bases in ℓ_1 and c_0. We make a detour from our route to solve both questions in the negative. For that we will need to apply the arguments that follow, keeping in mind that $\ell_1 = (c_0)^*$ is a GT-space (see [21]).

Proposition 3.9 *Let $\mathbb{X}$ be a quasi-Banach space, and let $\mathcal{X} = (\boldsymbol{x}_n)_{n=1}^\infty$ and $\mathcal{Y} = (\boldsymbol{x}_n^*)_{n=1}^\infty$ be sequences in $\mathbb{X}$ and $\mathbb{X}^*$, respectively. Suppose that $(\boldsymbol{x}_n, \boldsymbol{x}_n^*)_{n=1}^\infty$ is a biorthogonal system and that*

$$\boldsymbol{\varphi_u}[\mathcal{X}, \mathbb{X}](m)\, \boldsymbol{\varphi_u}[\mathcal{Y}, \mathbb{X}^*](m) \le Cm, \quad m \in \mathbb{N},$$

for some constant C. Then

$$\|\mathbb{1}_{\varepsilon,A}[\mathcal{X}, \mathbb{X}]\| \le C\|f\|,$$

for all $A \subseteq \mathbb{N}$ finite, all $\varepsilon \in \mathcal{E}_A$, and all $f \in \mathbb{X}$ such that $|\{n \in \mathbb{N} \colon |\boldsymbol{x}_n^*(f)| \ge 1\}| \ge |A|$.

Proof In the case when $\mathcal{X}$ spans the whole space $\mathbb{X}$, this proposition says that any bidemocratic basis is truncation quasi-greedy. In fact, the proof of [1, Proposition 5.7] gives this slightly more general result. □

Theorem 3.10 *Let $\mathbb{X}$ be a GT-space and let $\mathcal{X} = (\boldsymbol{x}_n)_{n=1}^\infty$ and $(\boldsymbol{x}_n^*)_{n=1}^\infty$ be sequences in $\mathbb{X}$ and $\mathbb{X}^*$, respectively. Suppose that $(\boldsymbol{x}_n, \boldsymbol{x}_n^*)_{n=1}^\infty$ is a biorthogonal system and that there is a constant C such that*

$$\|\mathbb{1}_{\varepsilon,A}[\mathcal{X}, \mathbb{X}]\| \le C\|f\| \tag{3.7}$$

whenever $A \subseteq \mathbb{N}$ and $f \in \mathbb{X}$ satisfy $|\boldsymbol{x}_n^*(f)| \ge 1 \ge |\boldsymbol{x}_k^*(f)|$ for $(n, k) \in A \times (\mathbb{N}\setminus A)$, and $\varepsilon = (\varepsilon_n)_{n\in A} \in \mathcal{E}_A$ is defined by $\boldsymbol{x}_n^*(f) = |\boldsymbol{x}_n^*(f)|\, \varepsilon_n$. Then, $\boldsymbol{\varphi_l}[\mathcal{X}, \mathbb{X}](m) \gtrsim m$ for $m \in \mathbb{N}$.

Proof In the case when $\mathcal{X}$ spans the whole space $\mathbb{X}$, this theorem says that any truncation quasi-greedy basis of a GT-space is democratic with fundamental function equivalent to $(m)_{m=1}^\infty$ (see [5, Theorem 4.3]). As a matter of fact, the proof of [5, Theorem 4.3] as well as the proofs of Lemmas 2.2 and 4.2 of [5] on which it relies, work for basic sequences $(\boldsymbol{x}_n)_{n=1}^\infty$ whose biorthogonal functionals extend to functionals $(\boldsymbol{x}_n^*)_{n=1}^\infty$ defined on the whole space $\mathbb{X}$ in such a way that condition (3.7) holds. □

Theorem 3.11 *Let $\mathcal{X}$ be a bidemocratic basis of a Banach space $\mathbb{X}$.*

(i) *If $\mathbb{X}$ is a GT-space, then $\mathcal{X}$ is equivalent to the canonical basis of ℓ_1.*
(ii) *If $\mathbb{X}^*$ is a GT-space, then $\mathcal{X}$ is equivalent to the canonical basis of c_0.*

Proof Suppose that $\mathbb{X}$ (resp., $\mathbb{X}^*$) is a GT-space. By Theorem 3.10 $\boldsymbol{\varphi_l}[\mathcal{X}, \mathbb{X}]$ (resp., $\boldsymbol{\varphi_l}[\mathcal{X}^*, \mathbb{X}^*]$) is equivalent to $(m)_{m=1}^\infty$. Hence, $\boldsymbol{\varphi_u}[\mathcal{X}^*, \mathbb{X}^*]$ (resp., $\boldsymbol{\varphi_u}[\mathcal{X}, \mathbb{X}]$) is bounded. This readily gives that $\mathcal{X}^*$ (resp. $\mathcal{X}$) is equivalent to the canonical basis of c_0. To conclude the proof of (i), we infer that $\mathcal{X}^{**}$ is equivalent to the canonical basis $\mathcal{B}_{\ell_1}$ of ℓ_1. Since $\mathcal{B}_{\ell_1}$ dominates $\mathcal{X}$ and $\mathcal{X}$ dominates $\mathcal{X}^{**}$ we are done. □

It is known that some results involving the TGA work for total bases but break down if we drop this assumption (see, e.g., [10, Theorem 4.2 and Example 4.5]). In view of this, another question springing from Theorem 3.8 is whether working with total bases makes a difference, i.e., whether bidemocratic total bases are quasi-greedy. We solve this question in the negative by proving the following theorem.

Theorem 3.12 *Let $1 < p < \infty$. Then any infinite-dimensional subspace of ℓ_p has a further subspace with a bidemocratic non-quasi-greedy total basis.*

Theorem 3.12 will follow as a consequence of the following general result.

Theorem 3.13 *Let $\boldsymbol{w} = (w_n)_{n=1}^\infty$ be a weight, and suppose that its primitive weight $(s_m)_{m=1}^\infty$ has the LRP and that $(s_m/m)_{m=1}^\infty$ is non-increasing. Let $\mathbb{X}$ be a Banach space with a total basis $\mathcal{X}$. Suppose that $\mathcal{X}$ is bidemocratic with $\boldsymbol{\varphi_u}[\mathcal{X}, \mathbb{X}](m) \approx s_m$ for $m \in \mathbb{N}$, and that $\mathcal{X}$ has a subsequence dominated by the unit vector basis of $d_{1,q}(\boldsymbol{w})$ for some $q > 1$. Then $\mathbb{X}$ has a subspace $\mathbb{Y}$ with a basis $\mathcal{Y}$ satisfying the following properties:*

(i) *$\mathcal{Y}$ is bidemocratic with $\boldsymbol{\varphi_u}[\mathcal{Y}, \mathbb{Y}](m) \approx s_m$ for $m \in \mathbb{N}$.*
(ii) *$\mathcal{Y}$ is total.*
(iii) *$\mathcal{Y}$ is not quasi-greedy.*
(iv) *$\mathcal{Y}$ is not Schauder in any order.*

Proof Choose a subsequence $\left(\boldsymbol{x}_{\eta(j)}\right)_{j=1}^\infty$ of $\mathcal{X} = (\boldsymbol{x}_n)_{n=1}^\infty$ so that $\mathbb{N}\backslash\eta(\mathbb{N})$ is infinite and the linear operator $T \colon d_{1,q}(\boldsymbol{w}) \to \mathbb{X}$ given by

$$T\left(\boldsymbol{e}_j\right) = \boldsymbol{x}_{\eta(j)}, \quad k \in \mathbb{N},$$

is bounded. Let $\psi \colon \mathbb{N} \to \mathbb{N}$ be the increasing sequence defined by $\psi(\mathbb{N}) = \mathbb{N}\backslash\eta(\mathbb{N})$. Since the harmonic series diverges we can recursively construct an increasing sequence $(t_k)_{k=0}^\infty$ of natural numbers with $t_0 = 0$ such that, if we put

$$\Lambda_k = H_{t_k} - H_{t_{k-1}},$$

then $\lim_k \Lambda_k = \infty$. For each $j \in \mathbb{N}$ define $\boldsymbol{y}_j = \boldsymbol{x}_{\eta(j)} + \boldsymbol{z}_j$, where

$$\boldsymbol{z}_j = \frac{s_j}{j}\boldsymbol{x}_{\psi(k)}, \quad k \in \mathbb{N}, \ t_{k-1} < j \le t_k.$$

It is clear that $(\boldsymbol{y}_j, \boldsymbol{x}^*_{\eta(j)})_{j=1}^\infty$ is a biorthogonal system. Thus, to see that $\mathcal{Y} := \left(\boldsymbol{y}_j\right)_{j=1}^\infty$ satisfies (i) it suffices to prove that, if $\mathcal{Z} = (\boldsymbol{z}_j)_{j=1}^\infty$, $\boldsymbol{\varphi_u}[\mathcal{Z}, \mathbb{X}](m) \lesssim s_m$ for $m \in \mathbb{N}$. Set $C_1 = \sup_n \|\boldsymbol{x}_n\|$. For every $A \subseteq \mathbb{N}$ with $|A| = m < \infty$ and $\varepsilon \in \mathcal{E}_A$ we have

$$\|\mathbb{1}_{\varepsilon,A}[\mathcal{Z}, \mathbb{X}]\| \le C_1 \sum_{j\in A} \frac{s_j}{j} \le C_1 \sum_{j=1}^m \frac{s_j}{j} \lesssim s_m.$$

Let us see that $\mathcal{Y}$ is a total basis of $\mathbb{Y} = [\mathcal{Y}]$. Set

$$\boldsymbol{z}^*_k = \boldsymbol{x}^*_{\psi(k)} - \sum_{j=1+t_{k-1}}^{t_k} \frac{s_j}{j}\boldsymbol{x}^*_{\eta(j)}, \quad k \in \mathbb{N}.$$

We have $\boldsymbol{z}_k^*(\boldsymbol{y}_j) = 0$ for all j and $k \in \mathbb{N}$. Therefore $\boldsymbol{z}_k^*(f) = 0$ for all $f \in \mathbb{Y}$ and $k \in \mathbb{N}$. Pick $f \in \mathbb{Y}$ and suppose that $\boldsymbol{x}_{\eta(j)}^*(f) = 0$ for all $j \in \mathbb{N}$. We infer that $\boldsymbol{x}_{\psi(k)}^*(f) = 0$ for all $k \in \mathbb{N}$. Since $\mathcal{X}$ is a total basis, $f = 0$.

To prove that $\mathcal{Y}$ is neither a quasi-greedy basis nor a Schauder basis under any reordering, we pick a permutation π of $\mathbb{N}$. For each $k \in \mathbb{N}$, choose $A_k \subseteq D_k := [1 + t_{k-1}, t_k] \cap \mathbb{N}$ minimal with the properties

$$l := \max(\pi^{-1}(A_k)) < \min(\pi^{-1}(D_k \setminus A_k)) \text{ and } \Gamma_k := \sum_{j \in A_k} \frac{1}{j} > \frac{\Lambda_k}{2}.$$

By construction,

$$\frac{\Lambda_k}{2} \geq \Gamma_k - \frac{1}{\pi(l)} \geq \Gamma_k - 1.$$

Then, if we set

$$\Theta_k := \sum_{j \in D_k \setminus A_k} \frac{1}{j} = \Lambda_k - \Gamma_k,$$

we have $\Gamma_k - \Theta_k = -\Lambda_k + 2\Gamma_k \in (0, 2]$. Also by construction, if we set

$$g_k = \sum_{j \in A_k} \frac{1}{s_j} \boldsymbol{y}_j, \quad h_k = \sum_{j \in D_k \setminus A_k} \frac{1}{s_j} \boldsymbol{y}_j, \quad k \in \mathbb{N},$$

then g_k is a partial-sum projection of $f_k := g_k - h_k$ with respect to the rearranged basis $(\boldsymbol{y}_{\pi(i)})_{i=1}^{\infty}$. Moreover, in the case when π is the identity map, g_k is a greedy projection of f_k. On one hand, if we set

$$f_k' = \sum_{j \in A_k} \frac{1}{s_j} \boldsymbol{e}_j - \sum_{j \in D_k \setminus A_k} \frac{1}{s_j} \boldsymbol{e}_j,$$

we have $f_k = T(f_k') + (\Gamma_k - \Theta_k) \boldsymbol{x}_{\psi(k)}$ for all $k \in \mathbb{N}$. By Lemma 3.4 (iii),

$$\|f_k'\|_{d_{1,q}(\boldsymbol{w})} = \left\| \sum_{j=1+t_{k-1}}^{t_k} \frac{1}{s_j} \boldsymbol{e}_j \right\|_{d_{1,q}(\boldsymbol{w})} \lesssim \max\{1, \Lambda_k^{1/q}\} \approx \Lambda_k^{1/q}.$$

Hence, $\|f_k\| \lesssim \Lambda_k^{1/q}$ for $k \in \mathbb{N}$. On the other hand, since $\boldsymbol{x}_{\psi(k)}^*(g_k) = \Gamma_k$, we have

$$\Lambda_k < 2\Gamma_k \leq 2C_2 \|g_k\|$$

where $C_2 = \sup_n \|\boldsymbol{x}_n^*\|$. Summing up,

$$\frac{\|g_k\|}{\|f_k\|} \gtrsim \Lambda_k^{1/q'} \xrightarrow[k\to\infty]{} \infty.$$

□

Corollary 3.14 *There is a bidemocratic total basis of ℓ_2 that is neither Schauder under any rearrangement of the terms nor quasi-greedy.*

Let us notice that the bases we construct to prove Theorem 3.12 are *not* Schauder bases. As the TGA does not depend on the particular way we reorder the basis, whereas being a Schauder basis does, studying the TGA within the framework of Schauder bases is somehow unnatural. Nonetheless, Schauder bases have provided a friendly framework to develop the greedy approximation theory with respect to bases since its beginning at the turn of the century. In fact, it is nowadays unknown even whether certain results involving the TGA work outside the framework of Schauder bases (see, e.g., [11])! Hence, in connection with our discussion it is natural to wonder whether bidemocratic Schauder bases are quasi-greedy. We close this section by providing a negative answer to this question too.

Theorem 3.15 *There is a Banach space with a bidemocratic Schauder basis which is not quasi-greedy.*

The proof of Theorem 3.15 relies on a construction that has its roots in [20], where it was used to build a conditional quasi-greedy basis. Variants of the original idea of Konyagin and Telmyakov have appeared in several papers with different motivations (see [1, 12, 18, 23]). Prior to tackling the proof we introduce a quantitative version of [1, Theorem 6.7].

Theorem 3.16 *Let $\mathcal{X}$ be a bidemocratic basis of a quasi-Banach space $\mathbb{X}$. Then*

$$\overline{\boldsymbol{g}}_m[\mathcal{X}^*, \mathbb{X}^*] \lesssim \overline{\boldsymbol{g}}_m[\mathcal{X}, \mathbb{X}], \quad m \in \mathbb{N}.$$

If, in addition, $\mathcal{X}$ is a Schauder basis and $\mathbb{X}$ is locally convex,

$$\overline{\boldsymbol{g}}_m[\mathcal{X}^*, \mathbb{X}^*] \approx \overline{\boldsymbol{g}}_m[\mathcal{X}, \mathbb{X}], \quad m \in \mathbb{N}.$$

Proof Given $D \subseteq \mathbb{N}$ finite, let S_D^* be the dual operator of $S_D = S_D[\mathcal{X}, \mathbb{X}]$. Let Δ_b be the bidemocracy constant of $\mathcal{X}$. Let $\mathbb{Y}$ be the closed subspace of $\mathbb{X}^*$ spanned by $\mathcal{X}^*$. Let $f^* \in \mathbb{Y}$, $m \in \mathbb{N}$, and B be a greedy set of f^* with $|B| = m$. Given $f \in \mathbb{X}$, we pick a greedy set $A = A_m(f)$ of $f \in \mathbb{X}$. The proof of [1, Theorem 6.7] gives

$$|S_B^*(f^*)(f) - f^*(S_A(f))| \le 2\Delta_b \|f\| \|f^*\|.$$

Therefore,

$$|S_B^*(f^*)(f)| \le (\overline{\boldsymbol{g}}_m[\mathcal{X}, \mathbb{X}] + 2\Delta_b)\|f\| \|f^*\|.$$

This yields the first estimate. To see the equivalence under the additional assumptions, we use that $\mathcal{X}^{**}$ is equivalent to $\mathcal{X}$ (see [6, Corollary 3.2.4]). □

Proposition 3.17 *Let $1 < p < \infty$. There is a Banach space $\mathbb{X}$ with a monotone Schauder basis $\mathcal{X}$ with the following properties:*

(i) For all finite sets $A \subseteq \mathbb{N}$ and all $\varepsilon \in \mathcal{E}_A$,

$$\|\mathbb{1}_{\varepsilon A}\| = |A|^{1/p} \quad \text{and} \quad \|\mathbb{1}^*_{\varepsilon A}\| = |A|^{1/p'},$$

where $1/p + 1/p' = 1$. Therefore, $\mathcal{X}$ is 1-bidemocratic.

(ii) Neither $\mathcal{X}$ nor $\mathcal{X}^*$ are quasi-greedy. Quantitatively,

$$\overline{\boldsymbol{g}}_m \approx \overline{\boldsymbol{g}}^*_m \approx \boldsymbol{k}_m \approx \boldsymbol{k}^*_m \approx (\log m)^{1/p'}, \quad m \in \mathbb{N}, \ m \geq 2.$$

Proof Put

$$\mathcal{D} := \{(m,k) \in \mathbb{N}^2 \colon 1 \leq k \leq m\},$$

where the elements are taken in the lexicographical order. Appealing to Lemma 3.2 we recursively construct a family $(r_{m,k}, s_{m,k})_{(m,k)\in\mathcal{D}}$ in $\mathbb{N}^2$ such that

$$m+1 < r_{m,k} < s_{m,k}, \quad 1 \leq k \leq m, \tag{3.8}$$

$$s_{m,k} < r_{m,k+1}, \quad 1 \leq k < m, \quad \text{and} \tag{3.9}$$

$$\frac{1}{k} - \frac{1}{m} \leq T_{m,k} := \sum_{j=r_{m,k}}^{s_{m,k}} \frac{1}{j} < \frac{1}{k}, \quad 1 \leq k \leq m. \tag{3.10}$$

Next, we choose a sequence $(A_m)_{m=1}^\infty$ of integer intervals contained in $\mathbb{N}$ so that $\max(A_m) < \min(A_{m+1})$ for all $m \in \mathbb{N}$, and

$$|A_m| = 2m + \sum_{k=1}^m s_{m,k} - r_{m,k}. \tag{3.11}$$

Note that we do not impose to the sets $(A_m)_{m\in\mathbb{N}}$ the condition that they form a partition of $\mathbb{N}$, so they are not uniquely determined by the family $(r_{m,k}, s_{m,k})_{(m,k)\in\mathcal{D}}$. Let

$$i_{m,k} = \min A_m + \sum_{j=1}^{k-1} \left(s_{m,j} - r_{m,j} + 2\right), \quad (m,k) \in \mathcal{D}.$$

Fix $m \in \mathbb{N}$. For each $n \in A_m$ there are unique integers $1 \leq k \leq m$ and $-1 \leq t \leq s_{m,k} - r_{m,k}$ so that $n = i_{m,k} + 1 + t$. Let us set

$$(d_{m,n}, \varepsilon_{m,n}) = \begin{cases} (k, 1) & \text{if } t = -1, \\ (r_{m,k} + t, -1) & \text{otherwise.} \end{cases}$$

Consider the subset of $\mathbb{N}$ given by

$$B_m = \{n \in A_m : \varepsilon_{m,n} = 1\} = \{i_{m,k} : 1 \le k \le m\}.$$

By definition, $(d_{m,n})_{n\in B_m}$ is an enumeration of the first m positive integers. In turn, the family $(d_{m,n})_{n\in A_m\setminus B_m}$, whose first element is $1 + r_{m,1}$, is increasing. Therefore, by (3.8),

$$\max_{n\in B_m} d_{m,n} < \min_{n\in A_m\setminus B_m} d_{m,n}, \tag{3.12}$$

and $(d_{m,n})_{n\in A_m}$ consists of distinct positive integers. Set $b_{m,n} = d_{m,n}^{-1/p'}$ for $m \in \mathbb{N}$ and $n \in A_m$. We infer that for each $m \in \mathbb{N}$ and $A \subseteq A_m$ we have

$$\sum_{n\in A} b_{m,n} \le \sum_{n=1}^{|A|} n^{-1/p'} \le p\,|A|^{1/p}, \quad \text{and} \tag{3.13}$$

$$\sum_{n\in A} b_{m,n}^{p'} \le H_{|A|}, \tag{3.14}$$

where, as before, H_m denotes the m-th harmonic number. Once the family $(b_{m,n})_{m\in\mathbb{N},n\in A_m}$ has been constructed, we define $\|\cdot\|_{\maltese}$ on c_{00} by

$$\left\|(a_n)_{n=1}^\infty\right\|_{\maltese} = \frac{1}{p} \sup_{\substack{m\in\mathbb{N}\\ l\in A_m}} \left| \sum_{\substack{n\in A_m\\ n\le l}} a_n b_{m,n} \right|.$$

Since $\max(A_m) < \min(A_{m+1})$ for all $m \in \mathbb{N}$, we have that $\|f\|_{\maltese} < \infty$ for all $f \in c_{00}$, so that $\|\cdot\|_{\maltese}$ is a semi-norm. Let $\mathbb{X}$ be the Banach space obtained as the completion of c_{00} endowed with the norm

$$\|f\| = \max\left\{\|f\|_p, \|f\|_{\maltese}\right\}.$$

It is routine to check that the unit vector system $\mathcal{X}$ is a monotone normalized Schauder basis of $\mathbb{X}$ whose coordinate functionals $\mathcal{X}^*$ are the canonical projections on each coordinate. It follows from (3.13) that

$$\|\mathbb{1}_{\varepsilon,A}\|_{\maltese} \le \frac{1}{p} \sup_{m\in\mathbb{N}} \sum_{n\in A\cap A_m} b_{m,n} \le |A|^{1/p}, \quad |A| < \infty,\ \varepsilon \in \mathcal{E}_A.$$

By definition, there is a norm-one linear map from $\mathbb{X}$ into ℓ_p which maps $\mathcal{X}$ to the unit vector system of ℓ_p. By duality, there is a norm-one map from $\ell_{p'}$ into $\mathbb{X}^*$ which maps the unit vector system of $\ell_{p'}$ to $\mathcal{X}^*$. In particular,

$$\|\mathbb{1}^*_{\varepsilon,A}\| \le |A|^{1/p'}, \quad |A| < \infty,\ \varepsilon \in \mathcal{E}_A.$$

We infer that (i) holds.

Define $a_{m,n} = \varepsilon_{m,n} d_{m,n}^{-1/p}$, so that $a_{m,n} b_{m,n} = \varepsilon_{m,n}/d_{m,n}$ for $m \in \mathbb{N}$ and $n \in A_m$. For each $m \in \mathbb{N}$ set

$$f_m = \sum_{n \in A_m} a_{m,n}\, \boldsymbol{x}_n.$$

Let $(m,k) \in \mathcal{D}$ and use the convention $i_{m,m+1} = 1 + \max(A_m)$. If $i_{m,k} \le l < i_{m,k+1}$ by construction we have

$$B_{m,k}(l) := \sum_{n=i_{m,k}}^{l} a_{m,n} b_{m,n} = \frac{1}{k} - \sum_{j=r_{m,k}}^{l-1+r_{m,k}-i_{m,k}} \frac{1}{j}.$$

Thus, the maximum and minimum values of $B_{m,k}(l)$ on the interval $i_{m,k} \le l < i_{m,k+1}$ are $1/k$ and $1/k - T_{m,k}$, respectively. Since by the right hand-side inequality in (3.10), $1/j - T_{m,j} > 0$ for all $1 \le j \le m$ we infer that

$$\|f_m\|_{\maltese} = \frac{1}{p} \max_{l \in A_m} \sum_{\substack{n \in A_m \\ n \le l}} a_{m,n} b_{m,n} = \frac{1}{p} \max_{1 \le k \le m} \frac{1}{k} + \sum_{j=1}^{k-1} \left(\frac{1}{j} - T_{m,j} \right).$$

Using the left hand-side inequality in (3.10) we obtain

$$\|f_m\|_{\maltese} \le \frac{1}{p} \max_{1 \le k \le m} \frac{1}{k} + \frac{k-1}{m} = \frac{1}{p}.$$

We also have

$$\|f_m\|_p^p = \sum_{k=1}^{m} \left(\frac{1}{k} + T_{m,k} \right) \le 2 H_m.$$

Hence, $\|f_m\| \le 2^{1/p} H_m^{1/p}$ for all $m \in \mathbb{N}$.

By (3.12), B_m is a greedy set of f_m. Since every coefficient of f_m is positive on B_m,

$$\|S_{B_m}(f_m)\| \ge \|S_{B_m}(f_m)\|_{\maltese} = \frac{1}{p} \sum_{j \in B_m} \frac{1}{d_{m,n}} = \frac{1}{p} H_m.$$

Summing up,

$$\frac{\|S_{B_m}(f_m)\|}{\|f_m\|} \ge \frac{1}{p\, 2^{1/p}} H_m^{1/p'}, \quad m \in \mathbb{N}.$$

Since $|B_m| = m$, this shows that $\boldsymbol{g}_m \ge p^{-1} 2^{-1/p} H_m^{1/p'}$ for all $m \in \mathbb{N}$.

By Theorem 3.16, it only remains to obtain the upper estimate for the unconditionality constants of $\mathcal{X}$. By (3.14) and Hölder's inequality, for all $A \subseteq \mathbb{N}$ with $|A| \le m$ we have

$$\|S_A(f)\|_{\maltese} \le \frac{1}{p}\|f\|_p \sup_m \left(\sum_{n\in A\cap A_m} |b_{m,n}|^{p'} \right)^{1/p'} \le \frac{1}{p} H_m^{1/p'} \|f\|_p.$$

Hence, $\boldsymbol{k}_m \le \max\{1, H_m^{1/p'}/p\}$ for all $m \in \mathbb{N}$. □

Remark 3.18 Given a basis $\mathcal{X}$ and an infinite subset $\mathbf{n}$ of $\mathbb{N}$, we say that $\mathcal{X}$ is $\mathbf{n}$-quasi-greedy if

$$\sup \left\{ \frac{\|S_A(f)\|}{\|f\|} : f \in \mathbb{X},\ A \ \text{ greedy set of } \ f,\ |A| \in \mathbf{n} \right\} < \infty$$

(see [23]). Note that the basis constructed in Proposition 3.17 is not $\mathbf{n}$-quasi-greedy for any increasing sequence $\mathbf{n}$.

Remark 3.19 The basis $\mathcal{X}$ in Proposition 3.17 has a subbasis isometrically equivalent to the unit vector basis of ℓ_p. Indeed, it is easy to check that $\left(\boldsymbol{x}_{i_{m,1}}\right)_{m=1}^{\infty}$ has this property. The basis $\mathcal{X}$ also has, as we next show, a block basis isometrically equivalent to the unit vector basis of c_0. Let $(A_m)_{m=1}^{\infty}$, $(B_m)_{m=1}^{\infty}$ and $(f_m)_{m=1}^{\infty}$ be as in that proposition, and define

$$g_m := S_{B_m}(f_m), \quad h_m = \frac{g_m}{\|g_m\|_{\maltese}}, \quad m \in \mathbb{N}.$$

Pick positive scalars $(\varepsilon_k)_{k=1}^{\infty}$ with $\sum_{k=1}^{\infty} \varepsilon_k^p = 1$. Since

$$\lim_m \frac{\|g_m\|_p}{\|g_m\|_{\maltese}} = 0,$$

there is a subsequence $\left(g_{m_k}\right)_{k=1}^{\infty}$ with $\left\|g_{m_k}\right\|_p \le \varepsilon_k \|g_{m_k}\|_{\maltese}$ for all $k \in \mathbb{N}$. Let $f = (a_k)_{k=1}^{\infty} \in c_{00}$. Since $\operatorname{supp}(h_m) \subseteq A_m$ for all m, we have

$$\left\| \sum_{k=1}^{\infty} a_k h_{m_k} \right\|_{\maltese} = \max_{k\in\mathbb{N}} |a_k| \|h_{m_k}\|_{\maltese} = \max_{k\in\mathbb{N}} |a_k|$$

and

$$\left\| \sum_{k=1}^{\infty} a_k h_{m_k} \right\|_p = \left(\sum_{k=1}^{\infty} |a_k|^p \|h_{m_k}\|_p^p \right)^{1/p} \le \left(\sum_{k=1}^{\infty} |a_k|^p \varepsilon_k^p \right)^{1/p} \le \max_{k\in\mathbb{N}} |a_k|.$$

Consequently, $\|\sum_{k=1}^{\infty} a_k h_{m_k}\| = \max_{k\in\mathbb{N}} |a_k|$.

4 Building Bidemocratic Conditional Quasi-greedy Bases

Probably, the most versatile method for building conditional quasi-greedy bases is the previously mentioned DKK-method due to Dilworth, Kalton and Kutzarova, which works only in the locally convex setting (i.e., for Banach spaces). It produces conditional almost greedy bases whose fundamental function either is equivalent to $(m)_{m=1}^{\infty}$ or has both the LRP and the URP. Thus, the DKK-method serves as a tool for constructing Banach spaces with bidemocratic conditional quasi-greedy bases whose fundamental function has both the LRP and the URP. In this section we develop a new method for building conditional bases that allows us to construct bidemocratic conditional quasi-greedy bases with an arbitrary fundamental function.

We write $\mathbb{X} \oplus \mathbb{Y}$ for the Cartesian product of the quasi-Banach spaces $\mathbb{X}$ and $\mathbb{Y}$ endowed with the quasi-norm

$$\|(f, g)\| = \max\{\|f\|, \|g\|\}, \quad f \in \mathbb{X},\ g \in \mathbb{Y}.$$

Given sequences $\mathcal{X} = (\boldsymbol{x}_n)_{n=1}^{\infty}$ and $\mathcal{Y} = (\boldsymbol{y}_n)_{n=1}^{\infty}$ in quasi-Banach spaces $\mathbb{X}$ and $\mathbb{Y}$ respectively, its direct sum is the sequence $\mathcal{X} \oplus \mathcal{Y} = (\boldsymbol{u}_n)_{n=1}^{\infty}$ in $\mathbb{X} \oplus \mathbb{Y}$ given by

$$\boldsymbol{u}_{2n-1} = (\boldsymbol{x}_n, 0), \quad \boldsymbol{u}_{2n} = (0, \boldsymbol{y}_n), \quad n \in \mathbb{N}.$$

If $\mathcal{X}$ and $\mathcal{Y}$ are bidemocratic bases, and $\boldsymbol{\varphi_u}[\mathcal{X}, \mathbb{X}] \approx \boldsymbol{\varphi_u}[\mathcal{Y}, \mathbb{Y}]$, then the basis $\mathcal{X} \oplus \mathcal{Y}$ of $\mathbb{X} \oplus \mathbb{Y}$ is also bidemocratic with

$$\begin{aligned}
\boldsymbol{\varphi_u}[\mathcal{X} \oplus \mathcal{Y}, \mathbb{X} \oplus \mathbb{Y}] &\approx \boldsymbol{\varphi_u}[\mathcal{X}, \mathbb{X}] \approx \boldsymbol{\varphi_u}[\mathcal{Y}, \mathbb{Y}],\\
\boldsymbol{g}_m[\mathcal{X} \oplus \mathcal{Y}, \mathbb{X} \oplus \mathbb{Y}] &= \max\{\boldsymbol{g}_m[\mathcal{X}, \mathbb{X}], \boldsymbol{g}_m[\mathcal{Y}, \mathbb{Y}]\},\\
\boldsymbol{k}_m[\mathcal{X} \oplus \mathcal{Y}, \mathbb{X} \oplus \mathbb{Y}] &= \max\{\boldsymbol{k}_m[\mathcal{X}, \mathbb{X}], \boldsymbol{k}_m[\mathcal{Y}, \mathbb{Y}]\}.
\end{aligned}$$

Loosely speaking, we could say that $\mathcal{X} \oplus \mathcal{Y}$ inherits naturally the properties of $\mathcal{X}$ and $\mathcal{Y}$. In contrast, "rotating"$\mathcal{X} \oplus \mathcal{Y}$ gives rise to more interesting situations. In this section we study the "rotated" sequence $\mathcal{X} \diamond \mathcal{Y} = (z_n)_{n=1}^{\infty}$ in $\mathbb{X} \oplus \mathbb{Y}$ given by

$$z_{2n-1} = \frac{1}{\sqrt{2}}(\boldsymbol{x}_n, \boldsymbol{y}_n), \quad z_{2n} = \frac{1}{\sqrt{2}}(\boldsymbol{x}_n, -\boldsymbol{y}_n), \quad n \in \mathbb{N}.$$

Note that

$$\sum_{n=1}^{\infty} a_n\, z_n = \frac{1}{\sqrt{2}}\left(\sum_{n=1}^{\infty}(a_{2n-1} + a_{2n})\boldsymbol{x}_n, \sum_{n=1}^{\infty}(a_{2n-1} - a_{2n})\boldsymbol{y}_n\right), \tag{4.1}$$

whenever the series converges.

To deal with bases built using this method, we introduce some notation. Given $A \subseteq \mathbb{N}$ we set

$$A^{\boldsymbol{o}} = \{2n-1 \colon n \in A\}, \quad A^{\boldsymbol{e}} = \{2n \colon n \in A\}.$$

Consider also the onto map $\eta\colon \mathbb{N}\to\mathbb{N}$ given by $\eta(n)=\lceil n/2\rceil$. Note that $\eta^{-1}(A)=A^{\boldsymbol{o}}\cup A^{\boldsymbol{e}}$ and $\eta(A^{\boldsymbol{o}})=\eta(A^{\boldsymbol{e}})=A$ for all $A\subseteq\mathbb{N}$.

Our first auxiliary result is pretty clear and well-known. In its statement we implicitly use the natural identification of $(\mathbb{X}\oplus\mathbb{Y})^*$ with $\mathbb{X}^*\oplus\mathbb{Y}^*$.

Lemma 4.1 (cf. [3, Theorem 2.6]) *Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are bases of $\mathbb{X}$ and $\mathbb{Y}$ respectively. Then $\mathcal{X}\diamond\mathcal{Y}$ is a basis of $\mathbb{X}\oplus\mathbb{Y}$ whose dual basis is $\mathcal{X}^*\diamond\mathcal{Y}^*$. Moreover, if $\mathcal{X}$ and $\mathcal{Y}$ are Schauder bases, so is $\mathcal{X}\diamond\mathcal{Y}$.*

Lemma 4.2 *Let $\mathcal{X}$ be a basis of a quasi-Banach space. There is a constant C such that*

$$\left\|\sum_{n=1}^{\infty} a_n\,\boldsymbol{x}_n\right\|\le C\boldsymbol{\varphi_u}(m)$$

whenever $|a_n|\le 1$ for all $n\in\mathbb{N}$ and $a_n\neq 0$ for at most m indices. Moreover, if $\mathbb{X}$ is p-Banach space, $0<p\le 1$, we can choose $C=(2^p-1)^{-1/p}$.

Proof It follows readily from [1, Corollary 2.3]. □

Lemma 4.3 *Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are bases of $\mathbb{X}$ and $\mathbb{Y}$ respectively. Then*

$$\boldsymbol{\varphi_u}[\mathcal{X}\diamond\mathcal{Y},\mathbb{X}\oplus\mathbb{Y}]\le C\max\{\boldsymbol{\varphi_u}[\mathcal{X},\mathbb{X}],\boldsymbol{\varphi_u}[\mathcal{Y},\mathbb{Y}]\}$$

for some constant C that only depends on the spaces $\mathbb{X}$ and $\mathbb{Y}$ (and it is $\sqrt{2}$ if $\mathbb{X}$ and $\mathbb{Y}$ are Banach spaces).

Proof Let $m\in\mathbb{N}$, $A\subseteq\mathbb{N}$ with $|A|\le m$, and $\varepsilon=(\varepsilon_n)_{n\in A}\in\mathcal{E}_A$. We extend ε by setting $\varepsilon_n=0$ if $n\in\mathbb{N}\backslash A$. Put

$$B=\{n\in\mathbb{N}\colon 2n-1\in A\}\cup\{n\in\mathbb{N}\colon 2n\in A\},$$

that is, $B=\eta(A)$. We have $|B|\le|A|$ and $|\varepsilon_{2n-1}\pm\varepsilon_{2n}|\le 2\chi_B(n)$ for all $n\in\mathbb{N}$. Thus, if C is the constant in Lemma 4.2,

$$\begin{aligned}
&\|\mathbb{1}_{\varepsilon,A}[\mathcal{X}\diamond\mathcal{Y},\mathbb{X}\oplus\mathbb{Y}]\|\\
&\quad=\frac{1}{\sqrt{2}}\max\left\{\left\|\sum_{n=1}^{\infty}(\varepsilon_{2n-1}+\varepsilon_{2n})\boldsymbol{x}_n\right\|,\left\|\sum_{n=1}^{\infty}(\varepsilon_{2n-1}-\varepsilon_{2n})\boldsymbol{y}_n\right\|\right\}\\
&\quad\le\frac{2C}{\sqrt{2}}\max\left\{\boldsymbol{\varphi_u}[\mathcal{X},\mathbb{X}](m),\boldsymbol{\varphi_u}[\mathcal{Y},\mathbb{Y}](m)\right\}.
\end{aligned}$$

□

Proposition 4.4 *Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are bidemocratic bases of quasi-Banach spaces $\mathbb{X}$ and $\mathbb{Y}$ respectively. Suppose also that*

$$s_m:=\boldsymbol{\varphi_u}[\mathcal{X},\mathbb{X}](m)\approx\boldsymbol{\varphi_u}[\mathcal{Y},\mathbb{Y}](m),\quad m\in\mathbb{N}.$$

Then $\mathcal{X} \diamond \mathcal{Y}$ is a bidemocratic basis of $\mathbb{X} \oplus \mathbb{Y}$. Moreover,

$$\boldsymbol{\varphi_u}[\mathcal{X} \diamond \mathcal{Y}, \mathbb{X} \oplus \mathbb{Y}](m) \approx s_m, \quad m \in \mathbb{N}.$$

Proof Since, by assumption,

$$\max\{\boldsymbol{\varphi_u}[\mathcal{X}^*, \mathbb{X}^*](m), \boldsymbol{\varphi_u}[\mathcal{Y}^*, \mathbb{Y}^*](m)\} \lesssim \frac{m}{s_m}, \quad m \in \mathbb{N},$$

applying Lemma 4.3 yields

$$\boldsymbol{\varphi_u}[\mathcal{X} \diamond \mathcal{Y}, \mathbb{X} \oplus \mathbb{Y}](m) \lesssim s_m, \quad m \in \mathbb{N},$$

and

$$\boldsymbol{\varphi_u}[\mathcal{X}^* \diamond \mathcal{Y}^*, \mathbb{X}^* \oplus \mathbb{Y}^*](m) \lesssim \frac{m}{s_m}, \quad m \in \mathbb{N}.$$

Using Lemma 4.1, these inequalities readily give the desired result. □

Proposition 4.5 *Let $\mathcal{X} = (\boldsymbol{x}_n)_{n=1}^\infty$ and $\mathcal{Y} = (\boldsymbol{y}_n)_{n=1}^\infty$ be non-equivalent bases of quasi-Banach spaces $\mathbb{X}$ and $\mathbb{Y}$ respectively. Then, $\mathcal{X} \diamond \mathcal{Y}$ is a conditional basis of $\mathbb{X} \oplus \mathbb{Y}$. Quantitatively, if*

$$\mathsf{c}_m = \{(a_n)_{n=1}^\infty \in \mathbb{F}^{\mathbb{N}} \colon 1 \le |\{n \in \mathbb{N} \colon a_n \neq 0\}| \le m\}$$

and

$$E_m[\mathcal{X}, \mathcal{Y}] = \sup_{(a_n)_{n=1}^\infty \in \mathsf{c}_m} \frac{\|\sum_{n=1}^\infty a_n\, \boldsymbol{x}_n\|}{\|\sum_{n=1}^\infty a_n\, \boldsymbol{y}_n\|}, \quad m \in \mathbb{N},$$

then

$$\boldsymbol{k}_m[\mathcal{X} \diamond \mathcal{Y}, \mathbb{X} \oplus \mathbb{Y}] \ge \frac{1}{2} \max\{E_m[\mathcal{X}, \mathcal{Y}], E_m[\mathcal{Y}, \mathcal{X}]\}, \quad m \in \mathbb{N}.$$

Proof Our proof relies on considering expansions relative to the rotated basis $\mathcal{X} \diamond \mathcal{Y}$ which define vectors whose first or second component is zero. Given $m \in \mathbb{N}$ and a sequence $(a_n)_{n=1}^\infty$ in c_m we set $A = \{n \in \mathbb{N} \colon a_n \neq 0\}$, and put

$$f_{\boldsymbol{o}} = \sum_{n \in A} a_n\, \boldsymbol{z}_{2n-1} \quad \text{and} \quad f_{\boldsymbol{e}} = \sum_{n \in A} a_n\, \boldsymbol{z}_{2n}.$$

By (4.1) we have

$$f_{\boldsymbol{o}} = \frac{1}{\sqrt{2}} \left(\sum_{n=1}^\infty a_n\, \boldsymbol{x}_n, \sum_{n=1}^\infty a_n\, \boldsymbol{y}_n \right),$$

$$f_{\boldsymbol{o}} + f_{\boldsymbol{e}} = \sqrt{2}\left(\sum_{n=1}^{\infty} a_n\, \boldsymbol{x}_n, 0\right), \quad \text{and}$$
$$f_{\boldsymbol{o}} - f_{\boldsymbol{e}} = \sqrt{2}\left(0, \sum_{n=1}^{\infty} a_n\, \boldsymbol{y}_n\right).$$

Therefore,

$$\frac{\|f_{\boldsymbol{o}}\|}{\|f_{\boldsymbol{o}} + f_{\boldsymbol{e}}\|} \geq \frac{1}{2}\frac{\|\sum_{n=1}^{\infty} a_n \boldsymbol{y}_n\|}{\|\sum_{n=1}^{\infty} a_n\, \boldsymbol{x}_n\|} \quad \text{and} \quad \frac{\|f_{\boldsymbol{o}}\|}{\|f_{\boldsymbol{o}} - f_{\boldsymbol{e}}\|} \geq \frac{1}{2}\frac{\|\sum_{n=1}^{\infty} a_n \boldsymbol{x}_n\|}{\|\sum_{n=1}^{\infty} a_n\, \boldsymbol{y}_n\|}.$$

Since $|A| \leq m$, these inequalities yield the desired lower estimate for $\boldsymbol{k}_m[\mathcal{X}\diamond\mathcal{Y}, \mathbb{X}\oplus\mathbb{Y}]$. $\square$

Proposition 4.5 gives that the conditionality constants of $\mathcal{X} \diamond \mathcal{Y}$ are bounded below by

$$\frac{1}{2}\max\left\{\frac{\boldsymbol{\varphi_u}[\mathcal{X}, \mathbb{X}]}{\boldsymbol{\varphi_u}[\mathcal{Y}, \mathbb{Y}]}, \frac{\boldsymbol{\varphi_u}[\mathcal{Y}, \mathbb{Y}]}{\boldsymbol{\varphi_u}[\mathcal{X}, \mathbb{X}]}, \frac{\boldsymbol{\varphi_l}[\mathcal{X}, \mathbb{X}]}{\boldsymbol{\varphi_l}[\mathcal{Y}, \mathbb{Y}]}, \frac{\boldsymbol{\varphi_l}[\mathcal{Y}, \mathbb{Y}]}{\boldsymbol{\varphi_l}[\mathcal{X}, \mathbb{X}]}\right\}.$$

Thus, applying our method to bases with non-equivalent fundamental functions yields "highly" conditional bases. In contrast, since bidemocratic bases are truncation quasi-greedy (see [1, Proposition 5.7]), a combination of Proposition 4.4 with Theorem 3.5 exhibits that we can apply the "rotation method" to bidemocratic bases with equivalent fundamental functions to obtain bases whose conditionality constants grow "slowly". However, the basis $\mathcal{X} \diamond \mathcal{Y}$ is always conditional unless $\mathcal{X}$ and $\mathcal{Y}$ are equivalent. In this context, since quasi-greedy bases are truncation quasi-greedy (see [1, Theorem 4.13]) we ask ourselves whether our construction preserves quasi-greediness. Our next result provides an affirmative answer to this question.

Theorem 4.6 *Let $\mathcal{X}$ and $\mathcal{Y}$ be bidemocratic bases of quasi-Banach spaces $\mathbb{X}$ and $\mathbb{Y}$ respectively. Suppose that*

$$\boldsymbol{\varphi_u}[\mathcal{X}, \mathbb{X}](m) \approx \boldsymbol{\varphi_u}[\mathcal{Y}, \mathbb{Y}](m), \quad m \in \mathbb{N}.$$

Then,

$$\boldsymbol{g}_m[\mathcal{X} \diamond \mathcal{Y}, \mathbb{X} \oplus \mathbb{Y}] \approx \max\{\boldsymbol{g}_m[\mathcal{X}, \mathbb{X}], \boldsymbol{g}_m[\mathcal{Y}, \mathbb{Y}]\}, \quad m \in \mathbb{N}.$$

In particular, $\mathcal{X} \diamond \mathcal{Y}$ is quasi-greedy if and only if $\mathcal{X}$ and $\mathcal{Y}$ are quasi-greedy.

Before the proof of Theorem 4.6 we give two auxiliary lemmas.

Lemma 4.7 *Let $\mathcal{X} = (\boldsymbol{x}_n)_{n=1}^{\infty}$ and $\mathcal{Y} = (\boldsymbol{y}_n)_{n=1}^{\infty}$ be bases of quasi-Banach spaces $\mathbb{X}$ and $\mathbb{Y}$ respectively. Suppose that $\mathcal{Y}$ is truncation quasi-greedy and that*

$$\boldsymbol{\varphi_u}[\mathcal{X}, \mathbb{X}](m) \lesssim \boldsymbol{\varphi_l}[\mathcal{Y}, \mathbb{Y}](m), \quad m \in \mathbb{N}.$$

Then, there is a constant C_0 such that

$$\left\| \sum_{n\in E} c_n\, \boldsymbol{x}_n \right\| \le C_0 \left\| \sum_{n=1}^{\infty} d_n\, \boldsymbol{y}_n \right\|$$

whenever $E \subseteq \mathbb{N}$ is finite and $\max_{n\in E} |c_n| \le \min_{n\in E} |d_n|$.

Proof It is immediate from Lemma 4.2 and Lemma 2.3 combined. □

Lemma 4.8 *Let $\mathcal{X}$ be a basis of a quasi-Banach space $\mathbb{X}$. If $\mathcal{X}$ is truncation quasi-greedy and democratic, then there is a constant C such that*

$$\overline{\boldsymbol{g}}_m[\mathcal{X}, \mathbb{X}] \le C\, \overline{\boldsymbol{g}}_k[\mathcal{X}, \mathbb{X}], \quad k \le m \le 2k.$$

In particular, the sequences $(\overline{\boldsymbol{g}}_m[\mathcal{X}, \mathbb{X}])_{m=1}^{\infty}$ and $(\boldsymbol{g}_m[\mathcal{X}, \mathbb{X}])_{m=1}^{\infty}$ are doubling.

Proof Let A be a greedy set of $f \in \mathbb{X}$ with $|A| = m$. Pick a greedy set B of f with $B \subseteq A$ and $|B| = k$. Since $|A\setminus B| \le |B|$, applying Lemma 4.7 with $\mathcal{X}$ and a permutation of $\mathcal{X}$ yields $\|S_{A\setminus B}(f)\| \le C_0 \|f - S_B\|$, where C_0 only depends on $\mathcal{X}$ and $\mathbb{X}$. Hence, if κ denotes the modulus of concavity of $\mathbb{X}$, $\|S_A(f)\| \le \kappa(C_0 + \overline{\boldsymbol{g}}_k)\|f\|$. □

Proof of Theorem 4.6 Let $(\boldsymbol{x}_n^*)_{n=1}^{\infty}$, $(\boldsymbol{y}_n^*)_{n=1}^{\infty}$, and $(\boldsymbol{z}_n^*)_{n=1}^{\infty}$ be the dual bases of $\mathcal{X} = (\boldsymbol{x}_n)_{n=1}^{\infty}$, $\mathcal{Y} = (\boldsymbol{y}_n)_{n=1}^{\infty}$, and $\mathcal{X} \diamond \mathcal{Y} = (\boldsymbol{z}_n)_{n=1}^{\infty}$, respectively. For $A \subseteq \mathbb{N}$, set $S_A = S_A[\mathcal{X} \diamond \mathcal{Y}, \mathbb{X} \oplus \mathbb{Y}]$, $S_A^{\mathbb{X}} = S_A[\mathcal{X}, \mathbb{X}]$, and $S_A^{\mathbb{Y}} = S_A[\mathcal{Y}, \mathbb{Y}]$. By Lemma 4.1 and (4.1),

$$\begin{aligned} S_{A^{\boldsymbol{e}} \cup A^{\boldsymbol{o}}}(g, 0) &= \sum_{n\in A} \boldsymbol{z}_{2n-1}^*(g, 0)\, \boldsymbol{z}_{2n-1} + \sum_{n\in A} \boldsymbol{z}_{2n}^*(g, 0)\, \boldsymbol{z}_{2n} \\ &= \sum_{n\in A} \boldsymbol{x}_n^*(g)\, \frac{\boldsymbol{z}_{2n-1} + \boldsymbol{z}_{2n}}{\sqrt{2}} \\ &= \sum_{n\in A} \boldsymbol{x}_n^*(g)\, (\boldsymbol{x}_n, 0) = (S_A^{\mathbb{X}}(g), 0) \end{aligned}$$

for all $g \in \mathbb{X}$ and all $A \subseteq \mathbb{N}$ finite. Similarly, for all $h \in \mathbb{Y}$ we have

$$\begin{aligned} S_{A^{\boldsymbol{e}} \cup A^{\boldsymbol{o}}}(0, h) &= \sum_{n\in A} \boldsymbol{z}_{2n-1}^*(0, h)\, \boldsymbol{z}_{2n-1} - \sum_{n\in A} \boldsymbol{z}_{2n}^*(0, h)\, \boldsymbol{z}_{2n} \\ &= \sum_{n\in A} \boldsymbol{y}_n^*(h)\, \frac{\boldsymbol{z}_{2n-1} - \boldsymbol{z}_{2n}}{\sqrt{2}} \\ &= \sum_{n\in A} \boldsymbol{y}_n^*(h)\, (0, \boldsymbol{y}_n) = (0, S_A^{\mathbb{Y}}(h)). \end{aligned}$$

Therefore, since $|A^{\boldsymbol{e}} \cup A^{\boldsymbol{o}}| = 2|A|$,

$$\boldsymbol{h}_m := \max\{\boldsymbol{g}_m[\mathcal{X}, \mathbb{X}], \boldsymbol{g}_m[\mathcal{Y}, \mathbb{Y}]\} \le \boldsymbol{g}_{2m}[\mathcal{X} \diamond \mathcal{Y}, \mathbb{X} \oplus \mathbb{Y}], \quad m \in \mathbb{N}.$$

Using Lemma 4.8, we obtain the desired upper estimate for $\boldsymbol{h}_m$.

Given a greedy set B of $f = (g, h) \in \mathbb{X} \oplus \mathbb{Y}$, let A_1, A_2 and A_{12} be disjoint subsets of $\mathbb{N}$ such that

$$B = (A_{12} \cup A_1)^{\boldsymbol{o}} \cup (A_{12} \cup A_2)^{\boldsymbol{e}}.$$

We have $|B| = 2|A_{12}| + |A_1| + |A_2|$. Set $A_0 = \mathbb{N} \setminus (A_{12} \cup A_1 \cup A_2)$. Let $(c_n)_{n=1}^\infty$ be the coefficients of f relative to $\mathcal{X} \diamond \mathcal{Y}$, let $(a_n)_{n=1}^\infty$ be the coefficients of g relative to $\mathcal{X}$, and let $(b_n)_{n=1}^\infty$ be the coefficients of h with respect to $\mathcal{Y}$. Notice that, by Lemma 4.1,

$$c_{2n-1} = \frac{1}{\sqrt{2}}(a_n + b_n), \quad c_{2n} = \frac{1}{\sqrt{2}}(a_n - b_n). \tag{4.2}$$

Set $c = \min\{|c_n| : n \in B\}$. The mere definition of the sets gives

(C.1) $|c_{2n-1}| \le c \le |c_{2n}|$ for all $n \in A_2^{\boldsymbol{o}}$,
(C.2) $|c_{2n}| \le c \le |c_{2n-1}|$ for all $n \in A_1^{\boldsymbol{e}}$,
(C.3) $\max\{|c_{2n-1}|, |c_{2n}|\} \le c$ for all $n \in A_0$, and
(C.4) $\max\{|c_{2n-1}|, |c_{2n}|\} \ge c$ for all $n \in \mathbb{N} \setminus A_0$.

Combining (4.2) with (C.3) gives $|a_n|, |b_n| \le \sqrt{2}c$ for all $n \in A_0$, i.e., $A_3 \cup A_4 \subseteq \mathbb{N} \setminus A_0$, where

$$A_3 = \{n \in \mathbb{N} : |a_n| > \sqrt{2}c\}, \quad A_4 = \{n \in \mathbb{N} : |b_n| > \sqrt{2}c\}.$$

Note that A_3 is a greedy set of g, A_4 is a greedy set of h, and

$$\max\{|A_3|, |A_4|\} \le |\mathbb{N} \setminus A_0| = |A_{12} \cup A_1 \cup A_2| \le |A_{12}| + |A_1| + |A_2| \le |B|.$$

Set $A_5 = \mathbb{N} \setminus (A_3 \cup A_0)$ and $A_6 = \mathbb{N} \setminus (A_4 \cup A_0)$. Taking into account that, for any $D \subseteq \mathbb{N}$, the coordinate projection on $\eta^{-1}(D)$ with respect to $\mathcal{X} \diamond \mathcal{Y}$ coincides with that with respect to the direct sum $\mathcal{X} \oplus \mathcal{Y}$ of bases $\mathcal{X}$ and $\mathcal{Y}$ we obtain

$$(S_{A_3}^{\mathbb{X}}(g), S_{A_4}^{\mathbb{Y}}(h)) - S_B(f) = S_{A_1^{\boldsymbol{e}}}(f) + S_{A_2^{\boldsymbol{o}}}(f) - (S_{A_5}^{\mathbb{X}}(g), S_{A_6}^{\mathbb{Y}}(h)).$$

Therefore, it suffices to prove that

$$\max\{\|S_{A_5}^{\mathbb{X}}(g)\|, \|S_{A_6}^{\mathbb{Y}}(h)\|, \|S_{A_1^{\boldsymbol{e}}}(f)\|, \|S_{A_2^{\boldsymbol{o}}}(f)\|\} \le C_1 \|f\|$$

for some constant C_1. Thus, the result would follow by applying the next two claims to the pairs of bases $(\mathcal{X}, \mathcal{Y})$, $(\mathcal{Y}, \mathcal{X})$, $(\mathcal{X}, \mathcal{Y}^-)$ and $(\mathcal{Y}^-, \mathcal{X})$ where $\mathcal{Y}^- = (-\boldsymbol{y}_n)_{n=1}^\infty$. □

Claim 1 There is a constant C such that

$$\left\| \sum_{n \in A} a_n \, \boldsymbol{x}_n \right\| \le C \left\| \sum_{n=1}^\infty a_n \, (\boldsymbol{x}_n, \boldsymbol{y}_n) + \sum_{n=1}^\infty b_n \, (\boldsymbol{x}_n, -\boldsymbol{y}_n) \right\|$$

whenever $A \subseteq \mathbb{N}$ is finite and $\max_{n \in A} |a_n| \le b := \min_{n \in A} |b_n|$.

Claim 2 There is a constant C such that

$$\left\|\sum_{n\in A} a_n\, \boldsymbol{x}_n\right\| \le C \left\|\left(\sum_{n=1}^\infty a_n\, \boldsymbol{x}_n, \sum_{n=1}^\infty b_n\, \boldsymbol{y}_n\right)\right\|$$

whenever $\max_{n\in A} |a_n| \le b := \min_{n\in A} \max\{|a_n+b_n|, |a_n-b_n|\}$.

Indeed, taking into account (C.1) and (C.2), applying Claim 1 would give

$$\left\|\sum_{n\in A_2^o} c_{2n-1}\boldsymbol{x}_n\right\| \le C \left\|\sum_{n=1}^\infty c_{2n-1}\,(\boldsymbol{x}_n, \boldsymbol{y}_n) + \sum_{n=1}^\infty c_{2n}\,(\boldsymbol{x}_n, -\boldsymbol{y}_n)\right\|,$$

$$\left\|\sum_{n\in A_2^o} c_{2n-1}\boldsymbol{y}_n\right\| \le C \left\|\sum_{n=1}^\infty c_{2n-1}\,(\boldsymbol{y}_n, \boldsymbol{x}_n) + \sum_{n=1}^\infty -c_{2n}\,(\boldsymbol{y}_n, -\boldsymbol{x}_n)\right\|,$$

$$\left\|\sum_{n\in A_1^e} c_{2n}\boldsymbol{x}_n\right\| \le C \left\|\sum_{n=1}^\infty c_{2n}\,(\boldsymbol{x}_n, -\boldsymbol{y}_n) + \sum_{n=1}^\infty c_{2n-1}\,(\boldsymbol{x}_n, \boldsymbol{y}_n)\right\|, \quad \text{and}$$

$$\left\|\sum_{n\in A_1^e} c_{2n}\boldsymbol{y}_n\right\| \le C \left\|\sum_{n=1}^\infty c_{2n}\,(-\boldsymbol{y}_n, \boldsymbol{x}_n) + \sum_{n=1}^\infty -c_{2n-1}\,(-\boldsymbol{y}_n, -\boldsymbol{x}_n)\right\|,$$

and these inequalities would yield $\|S_{A_2^o}(f)\| \le C\|f\|$ and $\|S_{A_1^e}(f)\| \le C\|f\|$. In turn, combining (4.2) with (C.4) gives

$$\begin{aligned} |a_n| &\le \max\{|a_n+b_n|, |a_n-b_n|\}, \quad n\in A_5,\\ |b_n| &\le \max\{|a_n+b_n|, |a_n-b_n|\} \quad n\in A_6. \end{aligned}$$

Therefore, applying Claim 2 would give

$$\|S_{A_5}^{\mathbb{X}}(f) = \left\|\sum_{n\in A_5} a_n\, \boldsymbol{x}_n\right\| \le C \left\|\left(\sum_{n=1}^\infty a_n\, \boldsymbol{x}_n, \sum_{n=1}^\infty b_n\, \boldsymbol{y}_n\right)\right\| = C\|f\|, \quad \text{and}$$

$$\|S_{A_6}^{\mathbb{Y}}(f) = \left\|\sum_{n\in A_6} b_n\, \boldsymbol{y}_n\right\| \le C \left\|\left(\sum_{n=1}^\infty b_n\, \boldsymbol{y}_n, \sum_{n=1}^\infty a_n\, \boldsymbol{x}_n\right)\right\| = C\|f\|.$$

Let us now prove Claim 1. Set $D_1 = \{n\in A \colon |a_n-b_n| \ge b\}$. If $n\in D_2 := A\setminus D_1$ then

$$|a_n+b_n| = |2b_n + (a_n-b_n)| \ge 2|b_n| - |a_n-b_n| > 2b - b = b.$$

Hence, if κ is the modulus of concavity of $\mathbb{X}$, applying Lemma 4.7 we obtain

$$\begin{aligned}\left\|\sum_{n\in A} a_n\, \boldsymbol{x}_n\right\| &\le \kappa\left(\left\|\sum_{n\in D_1} a_n\, \boldsymbol{x}_n\right\| + \left\|\sum_{n\in D_2} a_n\, \boldsymbol{x}_n\right\|\right)\\ &\le \kappa C_0\left(\left\|\sum_{n=1}^{\infty}(a_n-b_n)\, \boldsymbol{y}_n\right\| + \left\|\sum_{n=1}^{\infty}(a_n+b_n)\boldsymbol{x}_n\right\|\right)\\ &\le 2\kappa C_0 \max\left\{\left\|\sum_{n=1}^{\infty}(a_n+b_n)\, \boldsymbol{x}_n\right\|, \left\|\sum_{n=1}^{\infty}(a_n-b_n)\boldsymbol{y}_n\right\|\right\}\\ &= 2\kappa C_0\left\|\sum_{n=1}^{\infty} a_n\,(\boldsymbol{x}_n, \boldsymbol{y}_n) + \sum_{n=1}^{\infty} b_n\,(\boldsymbol{x}_n, -\boldsymbol{y}_n)\right\|.\end{aligned}$$

We conclude by proving Claim 2. Set $D_1 = \{n \in A\colon |a_n| \le |b_n|\}$ and $D_2 = A\setminus D_1$. Since

$$\max\{|a_n|, |b_n|\} \ge \frac{b}{2}, \quad n \in A,$$

we have $|b_n| \ge b/2$ for all $n \in D_1$ and $|a_n| \ge b/2$ for all $n \in D_2$. Therefore

$$\max_{n\in D_1} |a_n| \le 2 \min_{n\in D_1} |b_n|, \quad \max_{n\in D_2} |a_n| \le 2 \min_{n\in D_2} |a_n|.$$

Applying Lemma 4.7 we obtain

$$\begin{aligned}\left\|\sum_{n\in A} a_n\, \boldsymbol{x}_n\right\| &\le \kappa\left(\left\|\sum_{n\in D_1} a_n\, \boldsymbol{x}_n\right\| + \left\|\sum_{n\in D_2} a_n\, \boldsymbol{x}_n\right\|\right)\\ &\le \kappa C_0\left(\left\|\sum_{n=1}^{\infty} 2b_n\, \boldsymbol{y}_n\right\| + \left\|\sum_{n=1}^{\infty} 2a_n\, \boldsymbol{x}_n\right\|\right)\\ &\le 4\kappa C_0\left\|\left(\sum_{n=1}^{\infty} a_n\, \boldsymbol{x}_n, \sum_{n=1}^{\infty} b_n\, \boldsymbol{y}_n\right)\right\|.\end{aligned}$$

□

If φ is the fundamental function of a basis of a Banach space, then $(\varphi(m))_{m=1}^{\infty}$ and $(m/\varphi(m))_{m=1}^{\infty}$ are non-decreasing sequences (see [16]). Our next result says that any such φ corresponds in fact to a bidemocratic basis of a Banach space.

Theorem 4.9 *Let $(s_m)_{m=1}^{\infty}$ be a non-decreasing unbounded sequence of positive scalars. Suppose that $(m/s_m)_{m=1}^{\infty}$ is unbounded and non-decreasing. Then there is a Banach space $\mathbb{X}$ and a conditional bidemocratic quasi-greedy basis $\mathcal{X}$ of $\mathbb{X}$ whose fundamental function grows as $(s_m)_{m=1}^{\infty}$.*

Proof Let $\boldsymbol{w} = (w_n)_{n=1}^\infty$ denote the weight whose primitive weight is $(s_m)_{m=1}^\infty$. Then $d_{1,1}(\boldsymbol{w})$ is a Banach space whose dual space is the Marcinkiewicz space $m(\boldsymbol{w})$, consisting of all $f \in c_0$ whose non-increasing rearrangement $(a_n)_{n=1}^\infty$ satisfies

$$\|f\|_{m(\boldsymbol{w})} = \sup_m \frac{1}{s_m} \sum_{n=1}^m a_n < \infty$$

(see [13, Theorems 2.4.14 and 2.5.10]). Let $m_0(\boldsymbol{w})$ be the closed linear span of c_{00} in $m(\boldsymbol{w})$. Since the unit vector system is a boundedly complete basis of $d_{1,1}(\boldsymbol{w})$, an application of [6, Theorem 3.2.15] yields that the dual space of $m_0(\boldsymbol{w})$ is $d_{1,1}(\boldsymbol{w})$. In the language of bases, the dual basis of the unit vector system of $d_{1,1}(\boldsymbol{w})$ is the unit vector system of $m_0(\boldsymbol{w})$, and the other way around, i.e., the dual basis of the unit vector system of $m_0(\boldsymbol{w})$ is the unit vector system of $d_{1,1}(\boldsymbol{w})$. Moreover, the unit vector system of $m_0(\boldsymbol{w})$ is shrinking by [6, Theorem 3.2.17].

Let $\boldsymbol{w}^*$ be the weight whose primitive weight is $(m/s_m)_{m=1}^\infty$. We have the following chain of norm-one inclusions:

$$d_{1,1}(\boldsymbol{w}) \subseteq m_0(\boldsymbol{w}^*) \subseteq m(\boldsymbol{w}^*) \subseteq d_{1,\infty}(\boldsymbol{w}). \tag{4.3}$$

The right hand-side inclusion is clear. Let us prove the left hand-side inclusion. Let $(a_n)_{n=1}^\infty$ be the non-increasing rearrangement of $f \in c_0$. Given $m \in \mathbb{N}$ we define $(b_n)_{n=1}^\infty$ by $b_n = a_n$ is $n \le m$ and $b_n = 0$ otherwise. Using Abel's summation formula we obtain

$$\begin{aligned} \frac{s_m}{m} \sum_{n=1}^m a_n &= \frac{s_m}{m} \sum_{n=1}^\infty (b_n - b_{n+1}) n \\ &\le \sum_{n=1}^\infty (b_n - b_{n+1}) s_n \\ &= \sum_{n=1}^m a_n w_n \le \|f\|_{1,\boldsymbol{w}}. \end{aligned}$$

We infer from (4.3) that $d_{1,1}(\boldsymbol{w})$ and $m_0(\boldsymbol{w}^*)$ are Banach spaces for which the unit vector system is a symmetric basis with fundamental function $(s_m)_{m=1}^\infty$. Applying the rotation method with these bases yields a bidemocratic quasi-greedy basis of $d_{1,1}(\boldsymbol{w}) \oplus m_0(\boldsymbol{w}^*)$ with fundamental function equivalent to $(s_m)_{m=1}^\infty$.

To show that this basis is conditional, by Proposition 4.5 it suffices to show that $d_{1,1}(\boldsymbol{w})$ and $m_0(\boldsymbol{w}^*)$ are not isomorphic, so that $d_{1,1}(\boldsymbol{w}) \subsetneq m_0(\boldsymbol{w}^*)$. For that, we note that ℓ_1 is a complemented subspace of $d_{1,1}(\boldsymbol{w})$. Indeed, the proof in [7] works even without imposing to be non-increasing to $\boldsymbol{w}$. An appeal to [6, Theorem 3.3.1] concludes the proof. $\square$

Remark 4.10 Notice that in Theorem 4.9 we can obtain that $\mathcal{X}$ is 1-bidemocratic with $\boldsymbol{\varphi_u}[\mathcal{X}, \mathbb{X}](m) = s_m$ for all $m \in \mathbb{N}$. Indeed, if $(s_m)_{m=1}^\infty$ is a non-decreasing sequence of positive scalars such that $(m/s_m)_{m=1}^{\infty}$ is non-decreasing, and $\mathbb{X}$ is a p-Banach space, $0 < p \le 1$, with a bidemocratic basis $\mathcal{X}$ such that $\boldsymbol{\varphi_u}[\mathcal{X}, \mathbb{X}](m) \approx s_m$ for $m \in \mathbb{N}$, then, arguing as in the proof of [17, Theorem 2.1] (where unconditionality plays no role), we obtain an equivalent p-norm for $\mathbb{X}$ with respect to which $\boldsymbol{\varphi_u}[\mathcal{X}, \mathbb{X}](m) = s_m$ and $\boldsymbol{\varphi_u}[\mathcal{X}^*, \mathbb{X}^*](m) = m/s_m$ for all $m \in \mathbb{N}$.

Remark 4.11 In the case when $(s_m)_{m=1}^{\infty}$ has the URP we can give a more quantitative approach to the proof of Theorem 4.9. In this particular case we have $m(\boldsymbol{w}^*) = d_{1,\infty}(\boldsymbol{w})$. Applying the rotation method with the unit vector systems of $d_{1,p}(\boldsymbol{w})$ and $d_{1,q}(\boldsymbol{w})$, $0 < p < q \le \infty$, yields a bidemocratic quasi-greedy basis (of a locally r-convex quasi-Banach space, where $r = \min\{1, p\}$) whose fundamental function is equivalent to $(s_m)_{m=1}^{\infty}$. Combining (3.4) with Proposition 4.5 gives that the conditionality constants $(\boldsymbol{k}_m)_{m=1}^{\infty}$ of the basis we obtain satisfy

$$\boldsymbol{k}_m \gtrsim (H_m[\boldsymbol{w}])^{1/p-1/q}, \quad m \in \mathbb{N}.$$

In the particular case that $(s_m)_{m=1}^{\infty}$ has the LRP, by Lemma 3.4,

$$\boldsymbol{k}_m \gtrsim (\log m)^{1/p-1/q}, \quad m \in \mathbb{N}, \; m \ge 2.$$

Notice that, if $1 < p < q < \infty$ and $(s_m)_{m=1}^{\infty}$ has the LRP and the URP, then $\mathbb{X}$ is superreflexive [8, Theorem 3.16]. In particular, we find a bidemocratic quasi-greedy basis of a Banach space with $\boldsymbol{k}_m \gtrsim \log m$ for $m \ge 2$; and, for each $0 < s < 1$, a bidemocratic quasi-greedy basis of a superreflexive Banach space with $\boldsymbol{k}_m \gtrsim (\log m)^s$ for $m \ge 2$. Thus, the rotation method serves to built “highly conditional” almost greedy bases (see [2] for background on this topic).

Example 4.12 Let $\mathbb{X}$ be a Banach space with a greedy, non-symmetric basis $\mathcal{X}$ whose dual basis is also greedy. Then, if $\mathcal{X}_\pi$ is a permutation of $\mathcal{X}$ nonequivalent to $\mathcal{X}$, we have that $\mathcal{X} \diamond \mathcal{X}_\pi$ is a conditional quasi-greedy basis of $\mathbb{X} \oplus \mathbb{X}$. For instance, in light of [26, Theorem 2.1], this technique can be applied to the L_p-normalized Haar system to obtain a bidemocratic conditional quasi-greedy basis of $L_p([0, 1])$, $p \in (1, 2) \cup (2, \infty)$. Also, since, for the same values of p, the space ℓ_p has a greedy basis which is non-equivalent to the canonical basis (see [14, Theorem 2.1]), this technique yields a bidemocratic conditional basis of ℓ_p.

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